REPORT DOCUMENTATION PAGE

AFRL-SR-BL-TR-99-

0020

1. AGENCY USE ONLY (Leave blank)	2. REPORT DATE	3. REPORT TYPE AND DATES COVERED Progress Report,

4. TITLE AND SUBTITLE

Research on Relations Between Wavelets and Operators

5. FUNDING NUMBERS

F49620-96-1-0481

6. AUTHOR(S)

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UNC-Charlotte
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8. PERFORMING ORGANIZATION REPORT NUMBER

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)

Air Force Office of Scientific Research
110 Duncan Avenue, Suite B115
Bolling AFB, DC 20332

10. SPONSORING/MONITORING AGENCY REPORT NUMBER

11. SUPPLEMENTARY NOTES

12a. DISTRIBUTION/AVAILABILITY STATEMENT

Distribution Unlimited

12b. DISTRIBUTION CODE

D

13. ABSTRACT *(Maximum 200 words)*

During the period of the support, I have done some work related with my research, education of graduate students, and application of my results in industry which will be described this in four sections as follows:

1. Some Results and Publications.

2. Ph.D. Thesis directed.

3. Conferences and Presentations.

4. Transition to Industry.

14. SUBJECT TERMS

Waveltes, Operator Theory, Operator Algebras

15. NUMBER OF PAGES

16. PRICE CODE

17. SECURITY CLASSIFICATION OF REPORT	18. SECURITY CLASSIFICATION OF THIS PAGE	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT
UNCLASSIFIED	UNCLASSIFIED	UNCLASSIFIED	UL

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Scientific Report:

AFOSR, F49620-96-1-0481(1996-97)

–Research on relations between wavelets and operators–

By Xingde Dai

University of North Carolina at Charlotte

During the period of the support, I have done some work related with my research, education of graduate students, and application of my results in industry which will be illustrated in this report. I will describe this in four sections as follows:

1. Some Results and Publications.
2. Ph.D. Thesis directed.
3. Conferences and Presentations.
4. Transition to Industry.

1. Some Results and Publications.

(a) We (this principal investigator and his student Mr. Rufeng Liang) obtain the characterization of wavelet multipliers. This result was obtained independently (same time) by David Larson and his student Mr. Qing Gu. Here is the statement of the theorem:

Theorem 1 *A measurable unimodular function $\nu(s)$ has the property that for every orthogonal wavelet ψ, the function $\nu(s)\widehat{\psi}(s)$ is the Fourier Transform of an orthogonal wavelet if and only if the function $\frac{\nu(2s)}{\nu(s)}$ is 2π-periodic.*

Guido Weiss showed that the word "unimodular" can be deleted from the theorem. The function ν is called a wavelet multiplier.

(b) We obtain the characterization of phases of MRA wavelets. This result states as the following theorem.

Theorem 2 *Let ψ be an orthogonal wavelet with an MRA. Let η be a function defined by*

$$\widehat{\eta}(s) = e^{\frac{is}{2}}|\widehat{\psi}(s)|.$$

Then η is an orthogonal wavelet and there is a wavelet multiplier $\nu(s)$ such that

$$\widehat{\psi}(s) = \nu(s)\widehat{\eta}(s)$$

(c) We obtain the path-connectness of MRA wavelets. This result was obtained by Deguang Han, David Larson and Shijin Lu during the same week independently by using different techniques. The result state as the following theorem.

Theorem 3 *The set of MRA wavelets is path connected in $L^2(\mathbb{R})$-norm.*

(d) With David Larson and Darrin Speegle, this investigator obtains some new results on wavelets in $\mathbb{R}^n$

The results in a),b) and c) are collected in the following paper:

- (The Wutam Consortium,) *Basic Properties of Wavelets.* The Wutam Consortium is a group of people worked on the problems which is led by Guido Weiss and David Larson. The paper will appear in the Journal of Fourier Analysis and Applications.

The results in d) are materials in two papers:

- (with David Larson and Darrin Speegle) *Wavelet Sets in $\mathbb{R}^n$*, Journal of Fourier Analysis and Applications, pp. 451-456, Vol 3, No 4, 1997

- (with David Larson and Darrin Speegle) *Wavelet Sets in $\mathbb{R}^n$ II*, Contemporary Mathematics, to appear.

2. **Ph.D. Thesis directed.**

   Mr. Rufeng Liang enrolled into our graduate school in spring 1996. He took two of my courses on wavelets. He passed the preliminary exam in August, 1996 and passed the qualify exam and language exam (French) in spring 1998. Mr. Liang is writing a Ph.D. thesis. He has developed deep results in wavelet theory under the direction of this principal investigator. The results in a)-c) in the previous section are part of his thesis. Mr. Liang expects to defense his thesis in summer 1998. Mr. Liang received an Research Assistantship during spring and fall semesters, 1997. (Spring support by AFOSR, F49620-96-1-0481, fall support by DOE DE-AC22-94PC91008(BDM-OK)).

3. **Conferences and Presentations.**

   During summer 1997, this principal investigator was invited to give a talk at the International Conference On Operator Algebras and Operator Theory in Shanghai. He gave a presentation on the relation of wavelets and operators. During the trip he was invited to visit Bejing University, China. The following is the list of the author's talks during this supported (by AFOSR) period.

   - *An operator technique in wavelet theory,* International Conference On Operator Algebras and Operator Theory in Shanghai, East China Normal University, July, 1997. A 20 minutes talk. Invited.
   - *Wavelets and operator theory,* Beijing University, Beijing, China, June 17, 1997. A one hour talk. Invited.
   - *Applications of Wavelet Transforms in Petroleum Industry,* BDM-OK, Battlesville, Oklahoma, May 5, 1997.
   - *Some results in wavelet theory,* Functional Analysis conference at UNC-Charlotte, April, 1997. A 45 minutes talk. Invited.

4. Transition to Industry.

Innovative techniques and methodologies have been developed for upscaling reservoir properties using wavelet transforms. It has been shown that wavelet transform can be used to successfully upscale a 2-D reservoir permeability data under single and multiphase flow conditions. More recently, we developed a 3-D mathematical model for upscaling reservoir properties. It has also been demonstrated that wavelet transform, once coupled with geostatistics and fractal analysis, may provide a better approach for constructing complex geological models due to its unique ability to integrate multi-scale information.

The current techniques using wavelet transform for reservoir studies showed promising results under relatively ideal conditions. We conduct research for developing reliable algorithms and practical tools. The new techniques and algorithms will properly handle discontinuities, including the faults, fractures, geological facies contrasts and boundaries under upscaling context. In addition, we are developing a practical algorithm and related theories for effectively and precisely upscaling 3-D reservoir properties mapped in an irregular reservoir domain. We are developing a mathematical model to combine wavelet transform, geostatistics, and fractal analysis for efficiently integrating data of various scales in the characterization of the reservoir. This is a joint effort of this principal investigator and BDM-OK. This research is currently support by a grant provided by DOE.

Appendix

1. *Basic Properties of Wavelets*
2. *Wavelet Sets in* $\mathbb{R}^n$
3. *Wavelet Sets in* $\mathbb{R}^n$ *II*

Basic properties of wavelets

The Wutam Consortium

Dedicated to Eugene Fabes

September 12, 1997

Abstract

A wavelet multiplier is a function whose product with the Fourier transform of a wavelet is the Fourier transform of a wavelet. We characterize the wavelet multipliers, as well as the scaling function multipliers and low pass filter multipliers. We then prove that, if the set of all wavelet multipliers acts on the set of all MRA wavelets, the orbits are the sets of all MRA wavelets whose Fourier transforms have equal absolute values, and these are also equal to the sets of all MRA wavelets with the corresponding scaling functions having same absolute values of their Fourier transforms. As an application of these techniques we prove that the set of MRA wavelets is arcwise connected in $L^2(\mathbf{R})$.

Foreword

Two groups of collaborators, one led by Xingde Dai and David Larson and the other led by Eugenio Hernández and Guido Weiss, became aware that they were involved in the study of wavelets and obtained results that had much in common, even though different methods were employed. Early in 1996 we decided to exchange our ideas by holding meetings attended by us, our collaborators and students and by correspondence. This paper is what we hope is the first of a series of reports describing our joint results. Fourteen researchers are involved in this work. A few shorter notes describe more technical aspects of the work performed by different subgroups of this collection of fourteen researchers. At the end of this article, we shall describe the group that we call the "WUTAM CONSORTIUM". This is not the name of a Chinese city; it is the acronym for **Washington University, Texas A & M**, the two institutions associated with each of us.

1 Introduction

We shall begin by examining the most basic facts about orthonormal wavelets on the real line $\mathbf{R}$ and show that there is a partition of this set that is particularly useful for understanding wavelets and their properties. In particular, we will show that the wavelets, as a subset of the unit sphere in $L^2(\mathbf{R})$ have some unexpected connectivity properties; moreover, we will obtain several other properties of wavelets. In order to describe precisely the results we obtain, we must present appropriate definitions, notation and the facts that we assume the reader knows.

By an *orthonormal wavelet* (or, simply, a *wavelet*) we mean a function $\psi \in L^2(\mathbf{R})$ such that the system $\{\psi_{j,k}(x)\} = \{2^{j/2}\psi(2^j x - k)\}$, where j and k range through the integers $\mathbf{Z}$, forms an orthonormal basis of $L^2(\mathbf{R})$. There are two equations, involving the Fourier transform, $\hat{\psi}$, that completely characterize all wavelets. Let us agree to choose the following definition of the Fourier transform:

$$\hat{\psi}(\xi) = \int_{\mathbf{R}} \psi(x)e^{-i\xi x}dx$$

considered as a Lebesgue integral when $\psi \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$ and as a limit in the mean in the general case (see [11]). The characterization of wavelets is given by the following:

Proposition 1.1 *$\psi \in L^2(\mathbf{R})$ is an orthonormal wavelet if and only if $\|\psi\|_2 \geq 1$ and the following are true:*

(i) $\sum_{j\in\mathbf{Z}} |\hat{\psi}(2^j\xi)|^2 = 1$ a.e.;

(ii) $t_q(\xi) \equiv \sum_{j=0}^{\infty} \hat{\psi}(2^j\xi)\overline{\hat{\psi}(2^j(\xi + 2q\pi))} = 0$ a.e.

when q is an odd integer.

For a proof of this Proposition see chapter 7 of [6] where one can also find the appropriate references to G. Gripenberg and X. Wang who discovered this result independently.

These two equations can be used for constructing wavelets (and we shall come back to this later on); however, there is an elegant method that is often used to construct a large class of wavelets known as the *Multiresolution Analysis Method.* A *Multiresolution Analysis*, or, simply, an **MRA**, is a sequence $\mathcal{M} = \{V_j\}$, $j \in \mathbf{Z}$, of closed subspaces of $L^2(\mathbf{R})$ satisfying

(1.2) $\quad V_j \subseteq V_{j+1}$ for all $j \in \mathbf{Z}$

(1.3) $\quad f \in V_j$ if and only if $f(2(\cdot)) \in V_{j+1}$ for all $j \in \mathbf{Z}$

(1.4) $\quad \overline{\cup_{j\in\mathbf{Z}} V_j} = L^2(\mathbf{R})$

(1.5) There exists a function $\varphi \in V_0$ such that $\{\varphi(\cdot - k)\}$, $k \in \mathbf{Z}$, is an orthonormal basis for V_0.

It is not hard to show that properties (1.2), (1.3) and (1.5) imply

(1.6) $\quad \cap_{j\in\mathbf{Z}} V_j = \{0\}$

It follows readily from these properties that the spaces $W_j := V_{j+1} \cap V_j^{\perp}$, $j \in \mathbf{Z}$, are mutually orthogonal and their (orthogonal) direct sum equals $L^2(\mathbf{R})$:

$$L^2(\mathbf{R}) = \bigoplus_{j\in\mathbf{Z}} W_j \,. \tag{1}$$

It is clear from the above that a function $\psi \in W_0$ such that $\{\psi(\cdot - k)\}$, $k \in \mathbf{Z}$, is an orthonormal basis for W_0 is an orthonormal wavelet. The collection of all such wavelets will be referred to as the *set of all wavelets associated with the* **MRA** $\mathcal{M} = \{V_j\}$. We shall also say that an orthonormal wavelet ψ *is an* MRA *wavelet* if it is so associated with an MRA $\mathcal{M} = \{V_j\}$.

It is necessary that we review the well known explicit construction of such an MRA wavelet; this will allow us to make some observations that do not appear in the literature that form the basis for many of the results we will present. Given an MRA $\mathcal{M} = \{V_j\}$, let $\varphi \in V_0$ be a function satisfying (1.5). Such a function is called a *scaling function* associated with $\mathcal{M}$. Since $\frac{1}{2}\varphi(\frac{\cdot}{2}) \in V_{-1} \subseteq V_0$ (from (1.2), (1.3) and (1.5)) we can express this function in terms of the orthonormal basis $\{\varphi(\cdot + k)\}$, $k \in \mathbf{Z}$:

$$\frac{1}{2}\varphi\left(\frac{x}{2}\right) = \sum_{k\in\mathbf{Z}} \alpha_k \varphi(x+k) \ , \tag{2}$$

where

$$\alpha_k = \frac{1}{2}\int_{\mathbf{R}} \varphi(\frac{x}{2})\overline{\varphi(x+k)}dx$$

with $\sum_{k\in\mathbf{Z}} |\alpha_k|^2 < \infty$ and the convergence of the series in (2) is in the norm of $L^2(\mathbf{R})$. The Plancherel theorem shows that we can take the Fourier transform of both sides of equality (2) and obtain

$$\hat{\varphi}(2\xi) = \hat{\varphi}(\xi)\sum_{k\in\mathbf{Z}} \alpha_k e^{ik\xi} \equiv \hat{\varphi}(\xi)m(\xi) \ \ , \tag{3}$$

where

$$m(\xi) = \sum_{k\in\mathbf{Z}} \alpha_k e^{ik\xi}$$

is a 2π periodic function in $L^2(\mathbf{T}) = L^2([-\pi,\pi))$. This function m is uniquely determined by the scaling function φ via equation (3) and is called the *low pass filter* associated with φ. One can then show (see chapter 2 of [6]) that $\psi \in L^2(\mathbf{R})$ satisfying

$$\hat{\psi}(\xi) = e^{i\xi/2}\overline{m\left(\frac{\xi}{2}+\pi\right)}\hat{\varphi}\left(\frac{\xi}{2}\right) \tag{4}$$

is an orthonormal wavelet associated with $\mathcal{M}$. More generally, if

$$\hat{\psi}(\xi) = s(\xi)e^{i\xi/2}\overline{m\left(\frac{\xi}{2}+\pi\right)}\hat{\varphi}\left(\frac{\xi}{2}\right) \tag{5}$$

with s a *unimodular* (that is, $|s(\xi)| = 1$ a.e.) 2π periodic measurable function, then ψ is an orthonormal wavelet associated with $\mathcal{M}$. In fact, Proposition 2.13 in chapter 2 of [6] asserts ψ is an orthonormal wavelet associated with $\mathcal{M}$ if and only if (5) is satisfied. It is often useful to know if we can find a scaling function $\varphi \in V_0$ such that the simpler equality (4) is true (that is if (5) with $s(\xi) \equiv 1$ is satisfied). This is the content of the first theorem we shall prove:

Theorem I. *Suppose ψ is an orthonormal wavelet associated with an MRA $\mathcal{M} = \{V_j\}$, $j \in \mathbf{Z}$, then there exists a scaling function $\varphi \in V_0$ such that φ and ψ satisfy (4) where m is the low pass filter determined by φ .*

We have seen that the class of all wavelets can be characterized (see Proposition 1.1). There is also a simple (but not elementary) characterization of the subclass of all MRA wavelets; moreover, one can characterize all $\varphi \in L^2(\mathbf{R})$ that are scaling functions for an MRA. These characterizations are stated in the following two propositions.

Proposition 1.2 *An orthonormal wavelet ψ is an MRA wavelet if and only if*

$$D_\psi(\xi) \equiv \sum_{j=1}^{\infty} \sum_{k\in\mathbf{Z}} |\hat{\psi}(2^j(\xi + 2k\pi))|^2 = 1 \quad a.e.$$

Proposition 1.3 *A function $\varphi \in L^2(\mathbf{R})$ is a scaling function for an MRA if and only if*

(i) $\sum_{k\in\mathbf{Z}} |\hat{\varphi}(\xi + 2k\pi)|^2 = 1$ for a.e. $\xi \in \mathbf{R}$;

(ii) $\lim_{j\to\infty} |\hat{\varphi}(2^{-j}\xi)| = 1$ for a.e.$\xi \in \mathbf{R}$;

(iii) there exists a 2π periodic function $m \in L^2([-\pi, \pi))$ such that $\hat{\varphi}(2\xi) = m(\xi)\hat{\varphi}(\xi)$ for a.e. $\xi \in \mathbf{R}$.

The proofs for these results, as well as proper credits, can be found in chapter 7 of [6]. It is useful to point out that equality (i) in the last proposition is equivalent to the property that the sequence $\{\varphi(\cdot - k)\}$, $k \in \mathbf{Z}$, is an orthonormal system. The easy proof of this fact can also be found in [6].

An immediate consequence of Proposition 1.3 is that if φ is a scaling function, then the function $\tilde{\varphi}$ whose Fourier transform is $|\hat{\varphi}|$ is also a scaling function; however, $\tilde{\varphi}$ may be associated with a different MRA. An example of this phenomenon is provided by the Haar wavelet which is associated with the MRA determined by the scaling function φ that is the characteristic function of the interval $[-1, 0]$. Then

$$\hat{\varphi}(\xi) = e^{i\xi/2} \frac{\sin(\xi/2)}{(\xi/2)} .$$

Let $\tilde{\varphi} = (|\hat{\varphi}|)^\vee$. If $\tilde{\varphi}$ were a scaling function of this MRA, then $\hat{\tilde{\varphi}}(\xi) = l(\xi)\hat{\varphi}(\xi)$ with l a 2π periodic unimodular measurable function. Indeed, if $\tilde{\varphi}$, $\varphi \in V_0$ are each a scaling function for the MRA $\{V_j\}$, $j \in \mathbf{Z}$, then

$$\tilde{\varphi}(x) = \sum_{k\in\mathbf{Z}} \alpha_k \varphi(x - k)$$

with $\sum_{k\in\mathbf{Z}} |\alpha_k|^2 < \infty$ and the convergence is in the norm of $L^2(\mathrm{R})$. Taking Fourier transforms of both sides, we then have

$$\hat{\tilde{\varphi}}(\xi) = \left\{ \sum_{k\in\mathbf{Z}} \alpha_k e^{-ik\xi} \right\} \hat{\varphi}(\xi) \equiv l(\xi)\hat{\varphi}(\xi).$$

Since l is the Fourier series of a function in $L^2(\mathrm{T})$ it is 2π periodic. This fact and equality (i) in Proposition 1.3 imply that l is unimodular:

$$1 = \sum_{k\in\mathbf{Z}} |\hat{\tilde{\varphi}}(\xi + 2k\pi)|^2 = \sum_{k\in\mathbf{Z}} |l(\xi + 2k\pi)\hat{\varphi}(\xi + 2k\pi)|^2 =$$

$$|l(\xi)|^2 \sum_{k\in\mathbf{Z}} |\hat{\varphi}(\xi + 2k\pi)|^2 = |l(\xi)|^2 \cdot 1 = |l(\xi)|^2$$

almost everywhere. In the Haar wavelet situation we are considering, we would then have

$$l(\xi) = \{\mathrm{sgn}\xi\} \left\{ e^{-i\xi/2} \frac{|\sin(\xi/2)|}{\sin(\xi/2)} \right\} = e^{-i\xi/2}$$

for $\xi \in [-2\pi, 2\pi]\backslash\{0\}$, which shows that l cannot be 2π periodic.

It follows from our discussion that if φ_1 and φ_2 are scaling functions of the same MRA, then $|\hat{\varphi}_1(\xi)| = |\hat{\varphi}_2(\xi)|$ a.e. The same is true for two wavelets ψ_1 and ψ_2 associated with the same MRA: $|\hat{\psi}_1(\xi)| = |\hat{\psi}_2(\xi)|$. We have just seen that a scaling function whose Fourier transform equals $|\hat{\varphi}_1|$ (or $|\hat{\varphi}_2|$) may be associated with a different MRA and, as we shall presently see, the same is true for an MRA wavelet. We shall see that there are situations when it is desirable to reduce a problem involving wavelets to the case where the scaling function φ involved satisfies $\hat{\varphi}(\xi) \geq 0$. Toward this end, it is natural to study the properties of the class $\mathcal{S}_{\psi_0}$, ψ_0 an MRA wavelet, of all those MRA wavelets ψ such that $|\hat{\varphi}_0(\xi)| = |\hat{\varphi}(\xi)|$ a.e., where φ_0 is a scaling function associated with the same MRA as ψ_0 and φ is a scaling function related with ψ in the same way. It is not hard to see that $\mathcal{S}_{\psi_0}$ coincides with the class $\mathcal{W}_{\psi_0}$ of all those wavelets ψ such that $|\hat{\psi}_0(\xi)| = |\hat{\psi}(\xi)|$ a.e. (observe that it follows from Proposition 1.2 that any such ψ is also an MRA wavelet). Let us first observe that the inclusion $\mathcal{W}_{\psi_0} \subseteq \mathcal{S}_{\psi_0}$ is an immediate consequence of the equality

$$|\hat{\varphi}(\xi)|^2 = \sum_{j=1}^{\infty} |\hat{\psi}(2^j\xi)|^2 \qquad (6)$$

(see (2.16) in chapter 2 of [6]). Now suppose that $\psi \in \mathcal{S}_{\psi_0}$ and φ, φ_0 are scaling functions related to ψ and ψ_0 by equality (5). It is clear from (3) that the corresponding low pass filters m and m_0 have the same absolute value almost everywhere (recall that the low pass filter is uniquely determined by the scaling function). It then follows from (5) that $\hat{\psi}$ and $\hat{\psi}_0$ have absolute values that are equal a.e. ; this shows that $\mathcal{S}_{\psi_0} \subseteq \mathcal{W}_{\psi_0}$. Thus, whenever ψ_0 is an MRA wavelet,

$$\mathcal{W}_{\psi_0} = \mathcal{S}_{\psi_0}. \tag{7}$$

There is another description of this class of wavelets that is most useful to us. This involves the notion of a *wavelet multiplier* : a measurable function ν such that $(\nu\hat{\psi})^\vee$ is an orthonormal wavelet whenever ψ is an orthonormal wavelet. There is an elegant and simple characterization of these functions. Before announcing this result let us make a few observations. It will be of interest to us to consider what might be a larger class of such multipliers: those measurable functions ν such that $(\nu\hat{\psi})^\vee$ is an MRA wavelet whenever ψ is an MRA wavelet. It is natural to consider this notion of multipliers in connection with scaling functions and low pass filters: we say that ν is a *scaling function multiplier* if $\nu\hat{\varphi}$ is the Fourier transform of a scaling function whenever φ is a scaling function; similarly we say that μ is a *filter multiplier* provided μm is a low pass filter whenever m is a low pass filter. The characterization of these multipliers is the content of the second theorem that we shall prove:

Theorem II. *The class of wavelet multipliers coincides with both the class of MRA wavelet multipliers and the set of scaling function multipliers. Moreover, a measurable function ν belongs to any one of these classes if and only if it is unimodular and $\nu(2\xi)/\nu(\xi)$ is almost everywhere equal to a 2π periodic function. A measurable function μ is a filter multiplier if and only if μ is unimodular and is almost everywhere equal to a 2π periodic function.*

Let us now return to another description of the class $\mathcal{W}_{\psi_0}$. If ψ_0 is a wavelet let $\mathcal{M}_{\psi_0}$ be the collection of all wavelets ψ whose Fourier transfom equals $\nu\hat{\psi}_0$, where ν is a wavelet multiplier. Proposition 1.2 and the unimodularity of ν imply that any $\psi \in \mathcal{M}_{\psi_0}$ is an MRA wavelet when ψ_0 is an MRA wavelet. The third result we shall establish is

Theorem III. *If ψ_0 is an MRA wavelet, then $\mathcal{M}_{\psi_0} = \mathcal{W}_{\psi_0} = \mathcal{S}_{\psi_0}$.*

When we defined the class $\mathcal{M}_{\psi_0}$ we did not assume that ψ_0 is an MRA wavelet; moreover, the set $\mathcal{W}_{\psi_0}$ can also be defined whether or not ψ_0 is an MRA wavelet. Of course we must make this assumption in order to define the class $\mathcal{S}_{\psi_0}$; thus, we can only consider $\mathcal{M}_{\psi_0}$ and $\mathcal{W}_{\psi_0}$ when ψ_0 is not an MRA wavelet. These two sets are not always equal in this last case. This is one of the reasons we restrict our attention in this paper, for the most part, to MRA wavelets. Most of the rest of this work is devoted to applications of the equality obtained in Theorem III. Perhaps, the most interesting one is the fact that this last result can be used effectively for showing that the collection of MRA wavelets, as a subset of the unit sphere in $L^2(\mathbf{R})$ is arcwise connected. More precisely, we shall prove

Theorem IV. *If ψ_0 and ψ_1 are two MRA wavelets, then there exists a continuous map $A : [0,1] \longrightarrow L^2(\mathbf{R})$ such that $A(0) = \psi_0$, $A(1) = \psi_1$ and $A(t)$ is an MRA wavelet for all $t \in [0,1]$.*

If $f(\xi)$ is a complex valued function on $\mathbf{R}$, then $f(\xi) = e^{i\beta(\xi)}|f(\xi)|$ for some real valued function β (which is clearly not unique). We call such a function β a *phase* of the function f. Another application of Theorem III that we shall present is a characterization of the phases of the MRA wavelets. We will also make some observations about the phases of other wavelets. Finally, we will describe the "shorter notes" that accompany this paper.

2 The Proof of Theorem I

Given an MRA wavelet ψ satisfying equality (5) we must find a scaling function $\tilde{\varphi}$ with accompanying filter $\tilde{m}$ such that

$$\hat{\psi}(\xi) = e^{i\xi/2}\overline{\tilde{m}\left(\frac{\xi}{2}+\pi\right)}\hat{\tilde{\varphi}}\left(\frac{\xi}{2}\right) \tag{8}$$

where $\tilde{\varphi}$ and φ are scaling functions for the same MRA. But this means that $\hat{\tilde{\varphi}}(\xi) = t(\xi)\hat{\varphi}(\xi)$, with t a unimodular 2π periodic measurable function. A simple calculation shows that the filter m determined by φ is related to $\tilde{m}$

by the equality

$$\tilde{m}(\xi) = t(2\xi)\overline{t(\xi)}m(\xi). \tag{9}$$

Thus, equality (8) can be re-written in the form

$$\hat{\psi}(\xi) = e^{i\xi/2}\left\{\overline{t(\xi)}t\left(\frac{\xi}{2}\right)t\left(\frac{\xi}{2}+\pi\right)\right\}\overline{m\left(\frac{\xi}{2}+\pi\right)}\hat{\varphi}\left(\frac{\xi}{2}\right)$$

(we are using the 2π periodicity of t). Comparing this equality with (5) we see that we can reduce the proof of Theorem I to establishing the following Lemma:

Lemma 2.1 *Suppose s is a 2π periodic, unimodular and measurable function on* $\mathbf{R}$, *then there exists a 2π periodic, unimodular and measurable function t such that*

$$s(\xi) = \overline{t(\xi)}t\left(\frac{\xi}{2}\right)t\left(\frac{\xi}{2}+\pi\right) \tag{10}$$

Proof. We shall show that the solution $t(\xi)$ is completely determined by its restriction to the interval $[0,\pi)$ and this restriction can be an arbitrary unimodular measurable function. Toward this end let us rewrite equality (10) in the form

$$t(\xi) = \overline{t(\xi-\pi)}t(2(\xi-\pi))s(2\xi)\ , \tag{11}$$

where we have used the fact that the desired solution t is 2π periodic. Let us partition $[0,2\pi)$ into the subintervals $I_j = [2\pi(1-2^{-j}), 2\pi(1-2^{-j-1}))$, $j = 0,1,2,\ldots$; thus,

$$\bigcup_{j=0}^{\infty} I_j = [0,\pi) \cup [\pi, 3\pi/2) \cup [3\pi/2, 7\pi/4) \ldots = [0,2\pi)\ .$$

We begin by defining the restrictions, t_j, of t to the intervals I_j. Let us choose an arbitrary unimodular measurable function t_0 on $I_0 = [0,\pi)$. If $\xi \in I_1$ then $\xi-\pi$ and $2(\xi-\pi)$ belong to I_0. Hence, equality (11) defines the function t_1 on I_1 in terms of t_0 and s. In general, if $\xi \in I_{j+1}$ then $\xi-\pi \in I_0$ and $2(\xi-\pi)$ belongs to I_j . Thus, equality (11) defines t_{j+1} by letting

$$t_{j+1}(\xi) = \overline{t_0(\xi-\pi)}t_j(2(\xi-\pi))s(2\xi)$$

when $\xi \in I_{j+1}$. Having so defined t on $[0, 2\pi)$, we extend t to $\mathbf{R}$ by 2π periodicity. On the interval $[\pi, 2\pi)$ we rewrite(11) in the form

$$t(\xi)t(\xi - \pi) = t(2\xi)s(2\xi). \tag{12}$$

Using the 2π periodicity of t, let us observe that each side of (12) is a π periodic expression and, thus, this equality extends to $\cup_{k\in\mathbf{Z}}[k\pi, (k+1)\pi) = \mathbf{R}$. Then, putting $\eta = 2\xi$ and solving for $s(\eta)$ we obtain the desired equality (10) (let us observe that on several occasions here, and in the sequel, we use the fact that the reciprocal of an unimodular function is its complex conjugate). □

3 The Proof of Theorem II

We begin with the characterization of the filter multipliers. Suppose, first that μ satisfies the two conditions stated in the last part of the announcement of this theorem: it is unimodular and almost everywhere equal to a 2π periodic function. We claim that there exists a unimodular function t satisfying

$$\mu(\xi) = t(2\xi)\overline{t(\xi)} \tag{13}$$

(compare with equality (9) as a motivation for this equation). Let t be any measurable unimodular function defined on the set $S = [-2\pi, -\pi) \cup [\pi, 2\pi)$. If $\xi \in 2S$, let

$$t(\xi) = t\left(\frac{\xi}{2}\right)\mu\left(\frac{\xi}{2}\right)\ ; \tag{14}$$

then (13) is clearly satisfied for $\xi \in S$. We proceed inductively: assuming t is defined on $2^j S$, $j = 1, 2, \ldots, n$, so that (13) is valid on $\cup_{j=0}^{n-1} 2^j S$, we define $t(\xi)$ by equality (14) when $\xi \in 2^{n+1}S$. If $\xi \in 2^{-1}S$ we let $t(\xi) = t(2\xi)\overline{\mu(\xi)}$. We then use this last equality to proceed inductively for the definition of t on the sets $2^n S$, $n \leq -1$. Since $\cup_{n\in\mathbf{Z}} 2^n S = \mathbf{R}\backslash\{0\}$ we obtain a desired solution of (13) for almost every point of $\mathbf{R}$ (in this case, all $\xi \neq 0$).

Suppose that m is a low pass filter defined by the scaling function φ. We then claim that $\tilde{m} = \mu m$ is the low pass filter defined by the scaling function $\tilde{\varphi}$, where $\hat{\tilde{\varphi}} = t\hat{\varphi}$. That $\tilde{\varphi}$ is, indeed, a scaling function is an immediate consequence of Proposition 1.3: Properties (i) and (ii) are clearly satisfied since t is unimodular and they are satisfied by the scaling function φ; property (iii) is also satisfied since $\tilde{m} = \mu m$ is 2π periodic and

$$\hat{\tilde{\varphi}}(2\xi) = t(2\xi)\hat{\varphi}(2\xi) = t(2\xi)m(\xi)\hat{\varphi}(\xi) \;=\; t(2\xi)m(\xi)\overline{t(\xi)}t(\xi)\hat{\varphi}(\xi)$$
$$= \tilde{m}(\xi)\hat{\tilde{\varphi}}(\xi) \;.$$

Conversely, if μ is a filter multiplier, the fact that it must be unimodular is an immediate consequence of the fact that any low pass filter satisfies the equality

$$|m(\xi)|^2 + |m(\xi+\pi)|^2 = 1 \qquad (15)$$

for almost all ξ (see (2.5) in Chapter 2 of [6]). For the Haar wavelet, the low pass filter is $m_h(\xi) = (1+e^{i\xi})/2$ and for the *Shannon wavelet* the low pass filter is the 2π periodic function m_s that equals the characteristic function of the interval $[-\frac{\pi}{2}, \frac{\pi}{2})$ when restricted to $[-\pi, \pi)$ (see examples B and C in chapter 2 of [6]). The fact that μm_s, being a filter, must satisfy (15) implies that $|\mu(\xi)| = 1$ on the intervals $[\pi\frac{4k-1}{2}, \pi\frac{4k+1}{2})$, $k \in \mathbf{Z}$. This last property and the fact that μm_h must satisfy (15) imply that $|\mu(\xi)| = 1$ on $[\pi\frac{4k+1}{2}, \pi\frac{4k+3}{2})$, $k \in \mathbf{Z}$. Since the union of these two collections of intervals forms a partition of $\mathbf{R}$, we see that the filter multiplier μ must be unimodular.

Since $m_h(\xi) \neq 0$ when ξ is not an odd multiple of π and μm_h, being a low pass filter, equals a 2π periodic function m a.e., we conclude that $\mu(\xi) = m(\xi)/m_h(\xi)$ also equals a 2π periodic function a.e. This establishes the characterization of filter multipliers.

We now turn to the characterization of the wavelet multipliers. Let us first show that if ν is a unimodular function such that $\nu(2\xi)/\nu(\xi) = \nu(2\xi)\overline{\nu(\xi)}$ is 2π periodic, then it is a wavelet multiplier. We must show that $\hat{\tilde{\psi}} = \nu\hat{\psi}$ is the Fourier transform of a wavelet whenever ψ is a wavelet. We do this by showing that $\tilde{\psi}$ satisfies the properties (i), (ii) in Proposition 1.1 and $\|\tilde{\psi}\|_2 \geq 1$. Equality (i) and the last inequality are immediate since ν is unimodular and ψ satisfies these conditions (being a wavelet). Hence, the only thing needed is to show that

$$\sum_{j=0}^{\infty} \hat{\tilde{\psi}}(2^j\xi)\overline{\hat{\tilde{\psi}}(2^j(\xi+2q\pi))} = 0 \qquad (16)$$

a.e. when q is an odd integer. The summands of this series are equal to the products

$$\nu(2^j\xi)\overline{\nu(2^j(\xi+2q\pi))}\ \hat{\psi}(2^j\xi)\overline{\hat{\psi}(2^j(\xi+2q\pi))} \qquad (17)$$

for $j \geq 0$. When $j \geq 1$, using the unimodularity of ν and the 2π periodicity of $\nu(2\eta)\overline{\nu(\eta)}$, we have

$$\begin{aligned}\nu(2^j\xi)\overline{\nu(2^j(\xi+2q\pi))} &= \\ \nu(2^j\xi)\overline{\nu(2^{j-1}\xi)}\nu(2^{j-1}\xi)\overline{\nu(2^{j-1}(\xi+2q\pi))}&\cdot \\ \cdot\, \nu(2^{j-1}(\xi+2q\pi))\overline{\nu(2^j(\xi+2q\pi))} &= \\ \nu(2^{j-1}\xi)\overline{\nu(2^{j-1}(\xi+2q\pi))}\end{aligned} \tag{18}$$

since, by the 2π periodicity of $\nu(2\eta)\overline{\nu(\eta)}$ and the unimodularity of ν, the product of first two factors in (18) is the reciprocal of the product of the last two factors. We can continue this reduction and obtain

$$\nu(2^j\xi)\overline{\nu(2^j(\xi+2q\pi))} = \nu(\xi)\overline{\nu(\xi+2q\pi)}$$

when $j \geq 1$. Thus, the series in (16) equals

$$\nu(\xi)\overline{\nu(\xi+2q\pi)} \quad \sum_{j=0}^{\infty} \hat{\psi}(2^j\xi)\overline{\hat{\psi}(2^j(\xi+2q\pi))} = 0$$

a.e. and it follows that ν is, indeed, a wavelet multiplier.

If we show that a wavelet multiplier ν is unimodular, then Proposition 1.2 implies that $\tilde{\psi}$ is an MRA wavelet whenever ψ is an MRA wavelet. Consequently, the collection of wavelet multipliers is contained in the class of MRA wavelet multipliers. We shall show that if ν belongs to this last, possibly larger, class, then it does satisfy the two properties (unimodularity and the 2π periodicity of $\nu(2\xi)/\nu(\xi)$) we just used. Hence, the equality of these two classes will be established.

Let us assume, then, that ν is a wavelet multiplier and show that it must be unimodular. Let $F_n = \{\xi \in \mathbf{R} : |\nu(\xi)| \geq 1 + \frac{1}{n}\}$, $n \geq 1$. Let ψ be a wavelet such that $\{\xi : \hat{\psi}(\xi) = 0\}$ has measure 0 (for example, we can choose ψ to be the Haar wavelet). Then there exists an $\varepsilon > 0$ such that

$$|\{\xi : |\hat{\psi}(\xi)| > \varepsilon\} \cap F_n| \geq \frac{1}{2}|F_n|,$$

where $|S|$ denotes the Lebesgue measure of the measurable set S. Let $N \in \mathbf{N}$ be such that $\varepsilon(1+\frac{1}{n})^N > 1$. Then $\nu^N\hat{\psi}$ is the Fourier transform of a wavelet whose absolute value on the set $\{\xi : |\hat{\psi}(\xi)| > \varepsilon\} \cap F_n$ exceeds 1. But the

Fourier transform of a wavelet cannot exceed 1 on a set of positive measure; consequently, $|F_n| = 0$. Since $n \in \mathbb{N}$ is arbitrary, it follows that $|\nu(\xi)| \leq 1$ a.e. Moreover,

$$1 = \|\psi\|_2^2 = (1/2\pi) \int_{\mathbf{R}} |\hat{\psi}(\xi)|^2 d\xi \geq (1/2\pi) \int_{\mathbf{R}} |\nu(\xi)\hat{\psi}(\xi)|^2 d\xi = 1$$

because $(\nu\hat{\psi})^{\vee}$, being a wavelet, has L^2 norm 1. Therefore the last inequality is an equality and we can conclude that $|\nu(\xi)| = 1$ a.e. since $\hat{\psi}(\xi) \neq 0$ a.e. Let us observe that this argument also shows that an MRA wavelet multiplier must be unimodular.

We shall now show the 2π periodicity of $\nu(2\xi)/\nu(\xi)$ when ν is an MRA wavelet multiplier. It is easy to see that if ψ is an MRA wavelet then $e^{i\xi/2}|\hat{\psi}(\xi)|$ is the Fourier transform of an MRA wavelet. We can assume that ψ satisfies equality (5). Then, as we have already observed, $|\hat{\varphi}|$ is a scaling function whose associated low pass filter is $|m|$. Consequently,

$$e^{i\xi/2}\left| m\left(\frac{\xi}{2} + \pi\right) \hat{\varphi}\left(\frac{\xi}{2}\right) \right| = e^{i\xi/2}|\hat{\psi}(\xi)|$$

is the Fourier transform of an MRA wavelet. Let us choose $\psi = \psi_0$ so that $\hat{\psi}(\xi) \neq 0$ for a.e. ξ (again, the Haar wavelet provides us with such a wavelet). Let φ_0 be a scaling function, with filter m_0, such that

$$\hat{\psi}_0(\xi) = e^{i\xi/2}\, \overline{m_0\left(\frac{\xi}{2} + \pi\right)} \hat{\varphi}_0\left(\frac{\xi}{2}\right) \quad \text{and} \quad \hat{\psi}_1(\xi) = \nu(\xi) e^{i\xi/2}\, |\hat{\psi}_0(\xi)| \, .$$

Then ψ_1 is an MRA wavelet (by Proposition 1.2, since ν is unimodular) and, by Theorem I, we can find a scaling function φ_1 and accompanying filter m_1 such that

$$\nu(\xi) e^{i\xi/2} |\hat{\psi}_0(\xi)| = e^{i\xi/2}\, \overline{m_1\left(\frac{\xi}{2} + \pi\right)} \hat{\varphi}_1\left(\frac{\xi}{2}\right) \quad . \tag{19}$$

From (6) and the fact that $|\hat{\psi}_0(\xi)| = |\hat{\psi}_1(\xi)|$ we have $|\hat{\varphi}_0(\xi)| = |\hat{\varphi}_1(\xi)|$ and $|m_0(\xi)| = |m_1(\xi)|$ a.e. Let us now replace ξ in (19) by 2ξ and divide each side of this new equality by the corresponding sides in equality (19) to obtain

$$\frac{\nu(2\xi)\ |\hat{\psi}_0(2\xi)|}{\nu(\xi)\ |\hat{\psi}_0(\xi)|} \quad = \quad \frac{\overline{m_1(\xi + \pi)}\ \hat{\varphi}_1(\xi)}{\overline{m_1(\pi + \xi/2)}\ \hat{\varphi}_1(\xi/2)}$$

$$= \frac{\overline{m_1(\xi+\pi)}\, m_1(\xi/2)}{\overline{m_1(\pi+\xi/2)}} .$$

Thus,

$$\begin{aligned}\frac{\nu(2\xi)}{\nu(\xi)} &= \frac{|\hat{\psi}_0(\xi)|\,\overline{m_1(\xi+\pi)}\, m_1(\xi/2)}{|\hat{\psi}_0(2\xi)|\,\overline{m_1(\pi+\xi/2)}} = \\ &\frac{\overline{m_1(\xi+\pi)}}{|m_0(\xi+\pi)|}\,\frac{|m_0(\pi+\xi/2)|}{\overline{m_1(\pi+\xi/2)}}\,\frac{m_1(\xi/2)}{|m_0(\xi/2)|}\,,\end{aligned} \tag{20}$$

where we repeatedly used the expression of $\hat{\psi}$ in terms of m and $\hat{\varphi}$, as well as (3), to obtain the last equality; in particular, we have

$$\frac{|\hat{\psi}_0(\xi)|}{|\hat{\psi}_0(2\xi)|} = \frac{|m_0(\pi+\xi/2)|}{|m_0(\xi+\pi)|\,|m_0(\xi/2)|} .$$

The first fraction in (20) clearly defines a 2π periodic function. The product of the last two fractions in (20) equals

$$\frac{|m_1(\pi+\xi/2)|}{\overline{m_1(\pi+\xi/2)}}\,\frac{m_1(\xi/2)}{|m_1(\xi/2)|} = \frac{m_1(\pi+\xi/2)}{|m_1(\pi+\xi/2)|}\,\frac{m_1(\xi/2)}{|m_1(\xi/2)|}$$

which is clearly a 2π periodic function.

The characterization of scaling function multipliers is obtained in a similar, and simpler, way. If $t(\xi)$ is unimodular and $t(2\xi)\overline{t(\xi)}$ is 2π periodic it is immediate to verify that $t\hat{\varphi}$ satisfies the conditions of Proposition 1.3 when $\hat{\varphi}$ does. Thus, t is a scaling function multiplier. If, on the other hand, we know that t is a scaling function multiplier, the fact that it must be unimodular follows from the same argument we used for an MRA wavelet multiplier; this time we can use the Haar scaling function φ_0 instead of the Haar wavelet. We also use the fact that $\hat{\varphi}_0(\xi) \neq 0$ for almost all ξ to show the 2π periodicity of $\xi \to t(2\xi)\overline{t(\xi)}$. Since $t\hat{\varphi}_0 = \hat{\tilde{\varphi}}$ is the Fourier transform of a scaling function, $\hat{\tilde{\varphi}}(2\xi) = \tilde{m}(\xi)\hat{\tilde{\varphi}}(\xi)$, where $\tilde{m}$ is the filter associated with $\tilde{\varphi}$. But

$$\begin{aligned}\hat{\tilde{\varphi}}(2\xi) = t(2\xi)\hat{\varphi}_0(2\xi) &= t(2\xi)m_0(\xi)\hat{\varphi}_0(\xi) = \\ t(2\xi)m_0(\xi)\overline{t(\xi)}t(\xi)\hat{\varphi}_0(\xi) &= t(2\xi)\overline{t(\xi)}m_0(\xi)\hat{\tilde{\varphi}}(\xi) .\end{aligned}$$

It follows that $\tilde{m}(\xi) = t(2\xi)\overline{t(\xi)}m_0(\xi)$ and, consequently,

$$t(2\xi)\overline{t(\xi)} = \frac{\tilde{m}(\xi)}{m_0(\xi)}$$

(since the functions we are using are non-zero $a.e.$ and t is unimodular). But the last quotient is 2π periodic. □

Let us observe that the three types of multipliers we have characterized are related. An examination of our arguments shows that if $t(\xi)$ is a scaling function multiplier that produces the scaling function $\tilde{\varphi}$ from the scaling function φ, then $\mu(\xi) = t(2\xi)\overline{t(\xi)}$ is the multiplier that gives us the low pass filter $\tilde{m}$ defined by $\tilde{\varphi}$ from the low pass filter m associated with the scaling function φ. The low pass filters are uniquely determined, via equality (3), by the scaling functions. A low pass filter, however, can arise from different scaling functions. This is reflected by the fact that equation (13) has an infinitude of solutions. Each of these solutions is a scaling function multiplier.

If ν is a 2π periodic wavelet function multiplier, then each of the solutions t of equation (11) (with $\nu = s$) is a scaling function multiplier. There are several other relations between these multipliers and the roles they play in the structure of the sets $\mathcal{W}_{\psi_0}$, $\mathcal{S}_{\psi_0}$ and $\mathcal{M}_{\psi_0}$. These questions will be the subject of a future study.

4 The proof of Theorem III

We have already shown that $\mathcal{W}_{\psi_0} = \mathcal{S}_{\psi_0}$ (see equality (7)). Thus, to establish Theorem III we must show that $\mathcal{M}_{\psi_0}$ coincides with either $\mathcal{W}_{\psi_0}$ or $\mathcal{S}_{\psi_0}$. The inclusion $\mathcal{M}_{\psi_0} \subseteq \mathcal{W}_{\psi_0}$ is immediate since wavelet multiplier functions are unimodular.

We shall show that $\mathcal{S}_{\psi_0} \subseteq \mathcal{M}_{\psi_0}$. Suppose $\psi_1 \in \mathcal{S}_{\psi_0}$. Let φ_j be the scaling function such that

$$\hat{\psi}_j(\xi) = e^{i\xi/2}\,\overline{m_j\left(\pi + \frac{\xi}{2}\right)}\hat{\varphi}_j\left(\frac{\xi}{2}\right) \quad ,$$

$j = 0, 1$ (Theorem I guarantees the existence of these scaling functions and associated filters). As we have observed, $|\hat{\varphi}_0(\xi)| = |\hat{\varphi}_1(\xi)|$ a.e. Let $\tilde{\varphi}$ be the scaling function satisfying $\hat{\tilde{\varphi}} = |\hat{\varphi}_j(\xi)|$ (j is either 0 or 1); then $\tilde{m} = |m_j|$ is the associated low pass filter and let $\tilde{\psi}$ be the wavelet satisfying

$$\hat{\tilde{\psi}}(\xi) = e^{i\xi/2}\, \tilde{m}\left(\frac{\xi}{2} + \pi\right) \hat{\tilde{\varphi}}\left(\frac{\xi}{2}\right) \quad .$$

We shall show that there exists a wavelet multiplier function ν_j such that $\hat{\psi}_j = \nu_j \hat{\tilde{\psi}}$ for $j = 0$ and $j = 1$. Then $\nu = \nu_1 \overline{\nu_0}$ is also a wavelet multiplier function (this is immediate from the characterization given in Theorem II) and $\hat{\psi}_1 = \nu \hat{\psi}_0$. This, then, would establish the desired inclusion $\mathcal{S}_{\psi_0} \subseteq \mathcal{M}_{\psi_0}$.

In order to construct the wavelet multiplier ν_1 we first assume that

$$E = \{\xi : \hat{\varphi}_0(\xi) \neq 0\} = \{\xi : \hat{\tilde{\varphi}}(\xi) \neq 0\} = \{\xi : \hat{\varphi}_1(\xi) \neq 0\} = \mathbf{R} \ .$$

In this case

$$t(\xi) = \frac{\hat{\tilde{\varphi}}(\xi)}{\hat{\varphi}_1(\xi)}$$

is a well defined unimodular function. Moreover, it follows from our assumption and (3) that neither $m_1(\xi)$ nor $\tilde{m}(\xi)$ is ever 0. Let $\mu = \operatorname{sgn} m_1$ (that is, $\mu(\xi) = m_1(\xi)/\tilde{m}(\xi)$). We claim that

$$t(2\xi)\overline{t(\xi)} = \overline{\mu(\xi)} \ . \tag{21}$$

To establish this equality we first observe that

$$\begin{aligned} \hat{\tilde{\varphi}}(2\xi) = t(2\xi)\hat{\varphi}_1(2\xi) &= t(2\xi)m_1(\xi)\hat{\varphi}_1(\xi) = \\ t(2\xi)\overline{t(\xi)}m_1(\xi)t(\xi)\hat{\varphi}_1(\xi) &= t(2\xi)\overline{t(\xi)}m_1(\xi)\hat{\tilde{\varphi}}(\xi) \ . \end{aligned}$$

From this it follows that $\tilde{m}(\xi) = t(2\xi)\overline{t(\xi)}m_1(\xi)$ and (21) is an immediate consequence of this last equality. Moreover, $t(2\xi)\overline{t(\xi)} = \overline{\mu(\xi)}$, as the ratio of two nonzero 2π periodic functions, is also 2π periodic.

We claim that $\nu_1(\xi) = \overline{\mu(\pi + \xi/2)t(\xi/2)}$ is the desired wavelet multiplier function. It is clear that it is unimodular; furthermore, using (21), we have

$$\nu_1(2\xi)\overline{\nu_1(\xi)} = \overline{\mu(\pi+\xi)t(\xi)}\mu\left(\pi + \frac{\xi}{2}\right) t\left(\frac{\xi}{2}\right) = \overline{\mu(\pi+\xi)}\mu\left(\frac{\xi}{2}\right)\mu\left(\pi + \frac{\xi}{2}\right) \ .$$

But the last expression is 2π periodic since μ has this property. It follows from Theorem II that ν_1 is a wavelet multiplier function. Furthermore, we claim that

$$\hat{\psi}_1 = \nu_1 \hat{\tilde{\psi}} \ . \tag{22}$$

Indeed,

$$\hat{\psi}_1(\xi) = e^{i\xi/2}\,\overline{m_1\left(\pi + \frac{\xi}{2}\right)}\hat{\varphi}_1\left(\frac{\xi}{2}\right) =$$

$$e^{i\xi/2}\left\{\overline{m_1\left(\pi+\frac{\xi}{2}\right)}\mu\left(\pi+\frac{\xi}{2}\right)\right\}\left\{\overline{\mu\left(\pi+\frac{\xi}{2}\right)t\left(\frac{\xi}{2}\right)}\right\}\left\{t\left(\frac{\xi}{2}\right)\hat{\varphi}_1\left(\frac{\xi}{2}\right)\right\} =$$

$$e^{i\xi/2}\,\overline{\tilde{m}\left(\pi+\frac{\xi}{2}\right)}\nu_1(\xi)\hat{\tilde{\varphi}}\left(\frac{\xi}{2}\right) = \nu_1(\xi)\hat{\tilde{\psi}}(\xi) \ .$$

We have established Theorem III when $\hat{\varphi}_1(\xi) \neq 0$ for all ξ. When this is not the case, $t(\xi) = \hat{\tilde{\varphi}}(\xi)/\hat{\varphi}_1(\xi)$ is only defined on a set whose complement may have positive measure; when this is the case we need to find an appropriate extension of the unimodular function $t(\xi)$ for all $\xi \in \mathbf{R}$. We shall do this so that (21) is satisfied by this extension when μ is the 2π periodic unimodular function satisying $\mu(\xi)\tilde{m}(\xi) = m_1(\xi)$ and $\mu(\xi) = 1$ when $m_1(\xi) = 0$ (we can assume that $\hat{\tilde{\varphi}}(\xi) = |\hat{\varphi}_1(\xi)|$ and $\tilde{m}(\xi) = |m_1(\xi)|$ for all ξ). The argument that established (22) with $\nu_1(\xi) = \overline{\mu(\pi+\xi/2)t(\xi/2)}$ can then be carried out in order to finish the proof of Theorem III.

Thus, we shall now show how to construct this extension of $t(\xi)$. Let

$$F \equiv \{\xi \in \mathbf{R} : \hat{\varphi}_1(2^{l+1}\xi) = m_1(2^l\xi)\hat{\varphi}_1(2^l\xi) \text{ for all } l \in \mathbf{Z}\}.$$

If we let

$$F_l \equiv \{\xi \in \mathbf{R} : \hat{\varphi}_1(2^{l+1}\xi) = m_1(2^l\xi)\hat{\varphi}_1(2^l\xi)\}$$

for each $l \in \mathbf{Z}$, then $F = \cap_{l\in\mathbf{Z}} F_l$ and $\mathbf{R}\backslash F = \cup_{l\in\mathbf{Z}}(\mathbf{R}\backslash F_l)$. Observe that $F_l = 2^{-l}F_0$ and, since $|\mathbf{R}\backslash F_0| = 0$, we have $|\mathbf{R}\backslash F_l| = 0$ for all $l \in \mathbf{Z}$ and it follows that

$$|\mathbf{R}\backslash F| = 0 \ . \tag{23}$$

It is also immediate that

$$F = 2^l F \quad \text{for all } \ l \in \mathbf{Z}. \tag{24}$$

Let $E = \{\xi \in F : \hat{\varphi}_1(\xi) \neq 0\}$. If $2\xi \in E$ then, by the definition of E, $2\xi \in F$ and, by (24), we conclude that $\xi \in F$. In particular, $\hat{\varphi}_1(2\xi) = m_1(\xi)\hat{\varphi}_1(\xi)$; this equality and $\hat{\varphi}_1(2\xi) \neq 0$ (since $2\xi \in E$) imply $\hat{\varphi}_1(\xi) \neq 0$. Thus, $\xi \in E$; that is, $2^{-1}E \subseteq E$. It follows that

$$2^n E \subseteq 2^{n+1}E \quad n = 0, 1, 2, \dots \ . \tag{25}$$

Suppose that K is a measurable subset of $F \backslash \cup_{n \geq 0} 2^n E = \cap_{n \geq 0}(F \backslash 2^n E)$. We claim that

$$\chi_K(\xi)\hat{\varphi}_1(2^{-n}\xi) \equiv 0 \quad \text{for all } \ n \geq 0. \tag{26}$$

It is clear that this equality is true when $\xi \notin K$. If $\xi \in K$ then $\xi \in F$ and, by (24), $2^n\xi \in F$ for all $n \in \mathbf{Z}$; moreover, $2^{-n}\xi \notin E$ for all $n \geq 0$. Consequently, by the definition of E, $\hat{\varphi}_1(2^{-n}\xi) = 0$ for all $n \geq 0$. This proves (26).

The Fourier transform of $2^{n/2}\varphi_1(2^n x - k)$ is $2^{-n/2}e^{-ik2^{-n}\xi}\hat{\varphi}_1(2^{-n}\xi)$, k, $n \in \mathbf{Z}$. Hence, it follows from (26) that $\chi_K \perp \hat{V}_n$ for all $n \in \mathbf{Z}$. This fact, together with (2.4) imply that $|K| = 0$ and, consequently, $|F_n \backslash \cup_{n \geq 0} 2^n E| = 0$. Since

$$\mathbf{R} \backslash \cup_{n \geq 0} 2^n E \ \subseteq \ (\mathbf{R} \backslash F) \cup (F \backslash \cup_{n \geq 0} 2^n E) \ ,$$

the last equality and (23) imply

$$\left|\mathbf{R} \backslash \cup_{n \geq 0} 2^n E\right| = 0 \ . \tag{27}$$

We are now ready to construct the appropriate extension of $t(\xi)$. Let $\Delta_0 = E$ and $\Delta_n = 2^n E \backslash 2^{n-1} E$ for $n \geq 1$. Thus, if $H = \cup_{n \geq 0} 2^n E$, then

$$\mathbf{R} = (\mathbf{R} \backslash H) \cup (\cup_{n \geq 0} \Delta_n) \ ,$$

where $\Delta_m \cap \Delta_n = \emptyset$ when $m \neq n$, $(\mathbf{R} - H) \cap \Delta_n = \emptyset$ for $n \geq 0$ and $|\mathbf{R} - H| = 0$ (by (25) and (27)). Let μ be the 2π periodic, unimodular measurable function we defined in the paragraph that follows (22). We define $t_0(\xi)$, for $\xi \in \Delta_0 = E$ as the ratio $\hat{\tilde{\varphi}}(\xi)/\hat{\varphi}_1(\xi)$. Then, for $n \geq 1$ we define $t_{n+1}(\xi)$ for $\xi \in \Delta_{n+1}$ by letting

$$t_{n+1}(\xi) = \overline{\mu(\xi/2)} t_n(\xi/2)$$

assuming t_n is defined on Δ_n (observe that $\Delta_{n+1} = 2\Delta_n$ for $n \geq 1$). Then $t(\xi)$ is defined on $H = \cup_{n \geq 0} 2^n E = \cup_{n \geq 0} \Delta_n$ by putting $t(\xi) = t_n(\xi)$ when $\xi \in \Delta_n$. Since $\mathbf{R} \backslash H$ has measure 0 we only need to establish (21) on H; that is

$$\overline{t(2\xi)} t(\xi) = \mu(\xi) \tag{28}$$

for $\xi \in H$ with μ satisfying $\mu(\xi)\tilde{m}(\xi) = m_1(\xi)$ and $\mu(\xi) = 1$ when $m_1(\xi) = 0$. As was the case before, μ is a.e. equal to a 2π periodic measurable function. Also observe that $H = 2H$.

Suppose ξ and 2ξ belong to $\Delta_0 = E$. Then,

$$\hat{\tilde{\varphi}}(2\xi) = t_0(2\xi)\hat{\varphi}_1(2\xi) = t_0(2\xi)m_1(\xi)\hat{\varphi}_1(\xi) = t_0(2\xi)\overline{t_0(\xi)}m_1(\xi)\hat{\tilde{\varphi}}(\xi) .$$

Since $\hat{\tilde{\varphi}}(2\xi) = \tilde{m}(\xi)\hat{\tilde{\varphi}}(\xi)$ and all the values involved are non-zero, $\tilde{m}(\xi) = t_0(2\xi)\overline{t_0(\xi)}m_1(\xi)$ and we must have $\overline{t(2\xi)}t(\xi) = \overline{t_0(2\xi)}t_0(\xi) = \mu(\xi)$ in this case. Now suppose $\xi \in E$ but $2\xi \notin E$, so that $2\xi \in 2E\backslash E = \Delta_1$. Then the definition of $t = t_1$ on Δ_1 in terms of t_0 and μ show that (28) is satisfied for these values of ξ. We can now carry out an induction argument, similar to the ones we have already given, that shows that (28) is satisfied for all $\xi \in H$ as the disjoint union of the sets Δ_n. As we observed above, we can then establish (22) by the same argument we used above in order to finish the proof of Theorem III. □

5 The Proof of Theorem IV

The proof of this connectivity result will consist of two parts. The first is devoted to showing that the class $\mathcal{M}_{\psi_0}$ is arcwise connected. In the second part we show that if ψ_0 and ψ_1 are two MRA wavelets then an appropriate element of $\mathcal{M}_{\psi_0}$ can be connected by a continuous arc with a particular element of the class $\mathcal{M}_{\psi_1}$. It is clear that this, then, proves Theorem IV.

Suppose $\psi_1 \in \mathcal{M}_{\psi_0}$. Then $\hat{\psi}_1 = \nu\hat{\psi}_0$, where ν is a wavelet multiplier. Thus, by Theorem III

$$\mu(\xi) = \nu(2\xi)\overline{\nu(\xi)} \tag{29}$$

is unimodular and is equal a.e. to a 2π periodic function. For simplicity, let us assume that this holds for all ξ. We can write $\mu(\xi) = e^{i\beta(\xi)}$, where $0 \leq \beta(\xi) < 2\pi$ for all ξ. We thus obtain a unique 2π periodic phase β for the function μ. We can also write $\nu(\xi) = e^{i\lambda(\xi)}$. Our task is to find an appropriate choice for $\lambda(\xi)$ so that

$$\nu_t(\xi) \equiv e^{it\lambda(\xi)}, \tag{30}$$

$t \in [0,1]$, is a wavelet multiplier and the function $\theta : t \longrightarrow \psi_t(\xi) := \nu_t(\xi)\psi_0(\xi)$ is a continuous map from $[0,1]$ to $L^2(\mathbf{R})$ whose values are wavelets that, for $t = 0$ and $t = 1$, equal ψ_0 and ψ_1 respectively.

We do this by means of an argument that is a variant of one we used above (compare with the construction of the function $t(\xi)$ at the end of the last section). Let $E = [-4\pi, -2\pi) \cup [2\pi, 4\pi)$. On E we select λ so that $0 \leq \lambda(\xi) < 2\pi$. If $\xi \in 2E$ we let

$$\lambda(\xi) = \lambda(\xi/2) + \beta(\xi/2) \ . \tag{31}$$

Then, making use of (29), we have

$$e^{i\lambda(\xi)} = e^{i\lambda(\xi/2)}e^{i\beta(\xi/2)} = \nu(\xi/2)\mu(\xi/2) = \nu(\xi) \ .$$

We then continue inductively: assuming that $\lambda(\xi)$ is defined on 2^kE, $0 \leq k \leq n$, we let $\lambda(\xi)$, for $\xi \in 2^{n+1}E$, be defined by equality (31). In this way we have selected a phase $\lambda(\xi)$ of $\nu(\xi) = e^{i\lambda(\xi)}$ for $\xi \in \cup_{n\geq 0}2^nE$ in such a way that

$$\lambda(2\xi) - \lambda(\xi) = \beta(\xi) \ . \tag{32}$$

If $\xi \in 2^{-1}E$ we let $\lambda(\xi) = \lambda(2\xi) - \beta(\xi)$ and use this equality inductively to obtain $\lambda(\xi)$ defined on $\cup_{n\geq 1}2^{-n}E$. This gives us a function λ defined on $\mathbf{R}\backslash\{0\}$ satisfying (32) for all $\xi \neq 0$.

It follows from Theorem II that ν_t is a wavelet multiplier for each $t \in [0,1]$: it is clearly a unimodular function and, by (32),

$$\frac{\nu_t(2\xi)}{\nu_t(\xi)} = e^{it[\lambda(2\xi)-\lambda(\xi)]} = e^{it\beta(\xi)}$$

which is 2π periodic since β has this property. Thus, $\psi_t = \nu_t\psi_0 \in \mathcal{M}_{\psi_0}$.

Finally, we claim that $\theta(t) = \psi_t$ is a continuous mapping from $[0,1]$ to $L^2(\mathbf{R})$. This is the case since

$$|\hat{\psi}_t(\xi) - \hat{\psi}_s(\xi)|^2 = |e^{it\lambda(\xi)} - e^{is\lambda(\xi)}|^2 \, |\hat{\psi}_0(\xi)|^2 \ \leq \ 4|\hat{\psi}_0(\xi)|^2 \ ,$$

and, by the Lebesgue dominated convergence theorem and, then, Plancherel's theorem, we have, for each $s \in [0,1]$,

$$\lim_{t\to s} ||\psi_t - \psi_s||_2 = 0 \ .$$

This shows that each of the classes $\mathcal{M}_{\psi_0}$ is arcwise connected.

We now pass to the second part of the proof. Since, by Theorem III, $\mathcal{M}_{\psi_0} = \mathcal{S}_{\psi_0}$, to show that any two MRA wavelets are connected by a continuous path it suffices to choose ψ_0 to be the Shannon wavelet and select an appropriate element ψ_1 in any other class $\mathcal{S}_\psi$ where ψ is an MRA wavelet, and construct such a path that connects ψ_0 to ψ_1. The appropriate $\psi_1 \in \mathcal{S}_\psi$ is chosen so that it is associated with a scaling function φ_1 such that $\hat{\varphi}_1 \geq 0$. By the definition of $\mathcal{S}_\psi$ this can always be done and we can assume that

$$\hat{\psi}_1(\xi) = e^{i\xi/2}\,\overline{m_1(\pi + \xi/2)}\hat{\varphi}_1(\xi/2) \ ,$$

where m_1 is the low pass filter associated with φ_1. We recall that the Shannon wavelet ψ_0 satisfies the analogous equality

$$\hat{\psi}_0(\xi) = e^{i\xi/2}\,\overline{m_0(\pi + \xi/2)}\hat{\varphi}_0(\xi/2) \ ,$$

where $\hat{\varphi}_0(\xi) = \chi_{[-\pi,\pi)}(\xi)$ and the associated filter m_0 is the 2π periodic function that equals $\chi_{[-\pi/2,\pi/2)}$ on the period interval $[-\pi, \pi)$.

We shall construct a family of "intermediate" filters $\{m_t\}$, $t \in [0, 1]$, that connects m_0 with m_1 and then use these filters to define a family of scaling functions $\{\varphi_t\}$ from which we will obtain the desired path connecting ψ_0 to ψ_1. This family is defined on $[-\pi, \pi)$ by letting m_t satisfy

$$m_t(\xi) = \begin{cases} (1-t)m_0(\xi) + tm_1(\xi) & , \quad \xi \in [-\frac{\pi}{2}, \frac{\pi}{2}) \setminus [-(1-t)\frac{\pi}{2},\ (1-t)\frac{\pi}{2}) \\ 1 & , \quad \xi \in [-(1-t)\frac{\pi}{2},\ (1-t)\frac{\pi}{2}) \\ \sqrt{1 - m_t^2(\xi+\pi)} & , \quad \xi \in [-\pi, -\frac{\pi}{2}) \\ \sqrt{1 - m_t^2(\xi+\pi)} & , \quad \xi \in [\frac{\pi}{2}, \pi) \end{cases}$$

and, then, m_t is extended to $\mathbf{R}$ so that it is 2π periodic.

We thus, obtain a non-negative 2π periodic function m_t such that

$$m_t^2(\xi) + m_t^2(\xi + \pi) = 1 \quad \text{for all } \xi \in \mathbf{R} \text{ and } t \in [0, 1]. \tag{33}$$

Moreover, $m_t(\xi) \equiv 1$ on $[-(1-t)\frac{\pi}{2}, (1-t)\frac{\pi}{2})$. At $t = 0$ we obtain the Shannon filter and at $t = 1$ we have the filter defined by φ_1. Let

$$\hat{\varphi}_t(\xi) = \prod_{j=1}^{\infty} m_t(2^{-j}\xi)$$

for $\xi \in \mathbf{R}$. This product is well defined since $0 \leq m_t(\xi) \leq 1$. Moreover,

$$\hat{\varphi}_t(\xi) = 1 \quad \text{when } \xi \in [-(1-t)\pi, (1-t)\pi) \quad \text{and } 0 < t < 1. \tag{34}$$

It is also clear that $\hat{\varphi}_t(2\xi) = m_t(\xi)\hat{\varphi}_t(\xi)$ for all $\xi \in \mathbf{R}$. Thus, $\hat{\varphi}_t$ satisfies properties (ii)and (iii) of Proposition 1.3. We shall show that $\hat{\varphi}_t \in L^2(\mathbf{R})$ and its inverse Fourier transfom φ_t satisfies property (i) of this proposition. Thus φ_t is a scaling function. The argument we use to establish this is well known in wavelet theory (see, for example, the proof of Proposition 3.9 in Chapter 2 of [6]); however, our hypotheses are somewhat different and, for the sake of completeness, we include it here.

For $k \geq 1$ let

$$\mu_{t,k}(\xi) = \chi_{[-\pi,\pi)}(2^{-k}\xi) \prod_{j=1}^{k} m_t(2^{-j}\xi) \ .$$

For $n \in \mathbf{Z}$ and $k \geq 2$ let

$$I_{t,k}^n = \int_{\mathbf{R}} [\mu_{t,k}(\xi)]^2 e^{-in\xi} d\xi \ = \int_{-2^k\pi}^{2^k\pi} m_t^2(2^{-k}\xi) \prod_{j=1}^{k-1} m_t^2(2^{-j}\xi)\, e^{-in\xi} d\xi \ .$$

Observe that $\prod_{j=1}^{k-1} m_t^2(2^{-j}\xi)$ is $2^k\pi$ periodic. Hence, using this fact and (33) we have:

$$I_{t,k}^n = \int_0^{2^k\pi} [m_t^2(2^{-k}\xi) + m_t^2(2^{-k}\xi - \pi)] \prod_{j=1}^{k-1} m_t^2(2^{-j}\xi)\, e^{-in\xi} d\xi =$$

$$\begin{aligned}
\int_{-2^{k-1}\pi}^{2^{k-1}\pi} \prod_{j=1}^{k-1} m_t^2(2^{-j}\xi)\, e^{-in\xi} d\xi \ &= \ \int_{\mathbf{R}} [\mu_{t,k-1}(\xi)]^2\, e^{-in\xi} d\xi = \\
I_{t,k-1}^n \ &= \ \ldots = I_{t,1}^n = \\
\int_{\mathbf{R}} [\mu_{t,1}(\xi)]^2\, e^{-in\xi} d\xi \ &= \ \int_{-2\pi}^{2\pi} m_t^2(\xi/2)\, e^{-in\xi} d\xi = \\
\int_0^{2\pi} [m_t^2(\xi/2) + m_t^2(\xi/2 - \pi)]\, e^{-in\xi} d\xi \ &= \ \int_0^{2\pi} e^{-in\xi} d\xi = 2\pi\delta_{n,0} \ .
\end{aligned}$$

Clearly, $\lim_{k\to\infty} \mu_{t,k}(\xi) = \hat{\varphi}_t(\xi)$ for all $\xi \in \mathbf{R}$ and, by the above equalities, $I_{t,k}^0 = \|\mu_{t,k}\|_2^2 = 2\pi$. Thus, an application of Fatou's lemma shows that $\varphi_t \in L^2(\mathbf{R})$ with $\|\hat{\varphi}_t\|_2^2 \leq 2\pi$.

Suppose $\xi \in [-\pi, \pi)$, then $\xi/2 \in [-\pi/2, \pi/2)$ and, thus, $m_t(2^{-j}\xi) \geq 1 - t$ for $j \geq 1$. If $t < 1$ let us choose a k_0 sufficiently large so that $2^{-k_0} \leq 1 - t$.

Hence, if $\xi \in [-\pi, \pi)$ and $k \geq k_0$, it follows from (34) that $\hat{\varphi}_t(2^{-k}\xi) = 1$. Therefore,

$$\hat{\varphi}(\xi) = \prod_{j=1}^{k_0} m_t(2^{-j}\xi) \prod_{j=k_0+1}^{\infty} m_t(2^{-j}\xi) \geq$$
$$(1-t)^{k_0}\hat{\varphi}_t(2^{-k_0}\xi) = (1-t)^{k_0}$$

if $\xi \in [-\pi, \pi)$. Thus, we obtain

$$\chi_{[-\pi,\pi)}(\xi) \leq \frac{\hat{\varphi}_t(\xi)}{(1-t)^{k_0}} \tag{35}$$

for all $\xi \in \mathbf{R}$. From this we conclude that

$$\mu_{t,k}(\xi) = \chi_{[-\pi,\pi)}(2^{-k}\xi) \prod_{j=1}^{k} m_t(2^{-j}\xi) \leq$$

$$\frac{\hat{\varphi}_t(2^{-k}\xi)}{(1-t)^{k_0}} \prod_{j=1}^{k} m_t(2^{-j}\xi) = \frac{\hat{\varphi}_t(\xi)}{(1-t)^{k_0}} .$$

Since, as we just showed, $\hat{\varphi}_t \in L^2(\mathbf{R})$, we can apply Lebesgue's Dominated Convergence Theorem to obtain

$$\int_{\mathbf{R}} [\hat{\varphi}_t(\xi)]^2 e^{-in\xi} d\xi = \lim_{k\to\infty} \int_{\mathbf{R}} [\mu_{t,k}(\xi)]^2 e^{-in\xi} d\xi = 2\pi\delta_{n,o} .$$

Therefore, $\{\varphi_t(\cdot - k)\}$, $k \in \mathbf{Z}$ is an orthonormal system and we have shown that φ_t is a scaling function since it satisfies all the hypotheses of Proposition 1.3.

The proof of Theorem IV will be concluded once we show that $t \to \psi_t$ is a continuous mapping from $[0,1]$ to $L^2(\mathbf{R})$, where

$$\hat{\psi}_t(\xi) = e^{i\xi/2} m_t\left(\frac{\xi}{2} + \pi\right) \hat{\varphi}_t\left(\frac{\xi}{2}\right) .$$

We begin by examining the continuity of the map $t \longrightarrow \hat{\varphi}_t$. Let us fix a point $s \in [0,1]$ and list those points $\xi \in \mathbf{R}$ such that $t \longrightarrow m_t(\xi)$ is *not* continuous at s. If $0 < s < 1$ there are, at most, four such points within $[-\pi, \pi)$: $\pm\pi\frac{1-s}{2}$ and $\pm\pi\frac{1+s}{2}$ (when $s = 0$ or $s = 1$, there are, at most, two such points; moreover, for these two values of s, we are only interested in continuity from the right at 0 and from the left at 1). Because of the 2π periodicity of m_s we must also consider the $2k\pi$ translates of these points, $k \in \mathbf{Z}$. We now pass to the points of discontinuity at s of the map $t \longrightarrow m_t(2^{-j}\xi)$, $j \geq 1$. These values make up the set

$$\mathcal{D}_s = \left\{ \pm 2^j \pi \left(\frac{1-s}{2} + 2k \right), \ \pm 2^j \pi \left(\frac{1+s}{2} + 2l \right) : \quad k, l \in \mathbf{Z}, j \geq 1 \right\} .$$

Let $\mathcal{U} = \{\xi \in \mathbf{R} : \ \lim_{j\to\infty} \hat{\varphi}_1(2^{-j}\xi) = 1\}$. We then have $|\mathbf{R}\backslash\mathcal{U}| = 0 = |\mathcal{D}_s|$.

Lemma 5.1 *If $\xi \in \mathcal{U}\backslash\mathcal{D}_s$ then $\lim_{t\to s} \hat{\varphi}_t(\xi) = \hat{\varphi}_s(\xi)$.*

Proof. If $\eta \in [-\pi/2, \pi/2)$ then $m_t(\eta)$ is either 1 or $(1-t) + tm_1(\eta)$. In either case, $m_t(\eta) \geq m_1(\eta)$. From this it follows that

$$\hat{\varphi}_t(\eta) \geq \hat{\varphi}_1(\eta) \text{ when } \quad \eta \in [-\pi, \pi) . \tag{36}$$

For a fixed $\xi \in \mathcal{U}\backslash\mathcal{D}_s$ and $\varepsilon \in (0,1)$ we can find $M = M(\varepsilon, \xi)$ such that

$$\hat{\varphi}_1(2^{-n}\xi) \geq 1 - \varepsilon \quad \text{and } |2^{-n}\xi| < \pi$$

if $n \geq M$. Hence,

$$|\hat{\varphi}_t(\xi) - \hat{\varphi}_s(\xi)| = \left| \prod_{j=1}^{\infty} m_t\left(\frac{\xi}{2^j}\right) - \prod_{j=1}^{\infty} m_s\left(\frac{\xi}{2^j}\right) \right| =$$

$$\left| \left[\prod_{j=1}^{M} m_t\left(\frac{\xi}{2^j}\right)\right] \hat{\varphi}_t\left(\frac{\xi}{2^M}\right) - \left[\prod_{j=1}^{M} m_s\left(\frac{\xi}{2^j}\right)\right] \hat{\varphi}_s\left(\frac{\xi}{2^M}\right) \right| =$$

$$\left| \left[\hat{\varphi}_t\left(\frac{\xi}{2^M}\right) - 1\right] \prod_{j=1}^{M} m_t\left(\frac{\xi}{2^j}\right) - \left[\hat{\varphi}_s\left(\frac{\xi}{2^M}\right) - 1\right] \prod_{j=1}^{M} m_s\left(\frac{\xi}{2^j}\right) + \right.$$

$$\left. + \prod_{j=1}^{M} m_t\left(\frac{\xi}{2^j}\right) - \prod_{j=1}^{M} m_s\left(\frac{\xi}{2^j}\right) \right| \leq 2\varepsilon + \left| \prod_{j=1}^{M} m_t\left(\frac{\xi}{2^j}\right) - \prod_{j=1}^{M} m_s\left(\frac{\xi}{2^j}\right) \right| .$$

Because of our choice of ξ, $t \longrightarrow m_t(2^{-j}\xi)$ is continuous at s for each $j \geq 1$; therefore, there exists $\delta = \delta(\varepsilon)$ such that

$$\left| \prod_{j=1}^{M} m_t\left(\frac{\xi}{2^j}\right) - \prod_{j=1}^{M} m_s\left(\frac{\xi}{2^j}\right) \right| < \varepsilon$$

if $|t - s| < \delta$. Hence, $|\hat{\varphi}_t(\xi) - \hat{\varphi}_s(\xi)| < 3\varepsilon$ if $|t - s| < \delta$ and the lemma is proved. □

Since $t \longrightarrow m_t(\xi/2 + \pi)$ is continuous at s whenever $\xi/2 + \pi$ is not a translate by $2k\pi$, $k \in \mathbf{Z}$, of the points of the form $\pm\pi(1 \pm s)/2$ and $\xi \in \mathcal{U}$ if and only if $\xi/2 \in \mathcal{U}$ it follows from Lemma 5.1 that

$$\lim_{t \longrightarrow s} \hat{\psi}_t(\xi) = \hat{\psi}_s(\xi) \tag{37}$$

for a.e. $\xi \in \mathbf{R}$. Since $\|\hat{\psi}_t\|_2^2 = 2\pi = \|\hat{\psi}_s\|_2^2$ and m_t, m_s, $\hat{\varphi}_t$, and $\hat{\varphi}_s$ are non negative, an application of Schwarz's inequality gives us

$$0 \leq \int_{\mathbf{R}} m_t\left(\frac{\xi}{2} + \pi\right) \hat{\varphi}_t\left(\frac{\xi}{2}\right) m_s\left(\frac{\xi}{2} + \pi\right) \hat{\varphi}_s\left(\frac{\xi}{2}\right) d\xi = \langle \hat{\psi}_t, \hat{\psi}_s \rangle \leq 2\pi \ .$$

We claim that

$$\lim_{t \to s} \langle \hat{\psi}_t, \hat{\psi}_s \rangle = 2\pi \ . \tag{38}$$

If this were not the case, there would exist a sequence $\{t_n\} \subseteq [0,1]$ and an $\varepsilon > 0$ such that $\lim_{n \to \infty} t_n = s$ and

$$\int_{\mathbf{R}} m_{t_n}\left(\frac{\xi}{2} + \pi\right) \hat{\varphi}_{t_n}\left(\frac{\xi}{2}\right) m_s\left(\frac{\xi}{2} + \pi\right) \hat{\varphi}_s\left(\frac{\xi}{2}\right) d\xi \leq 2\pi - \varepsilon$$

for $n = 1, 2, 3, \ldots$. Let us choose a compact subset K of $\mathbf{R}$ such that

$$\int_K |\hat{\psi}_s(\xi)|^2 d\xi > 2\pi - \varepsilon \ .$$

Then, by (37) and the Dominated Convergence theorem (recall that $|\hat{\psi}(\xi)| \leq 1$ when ψ is a wavelet) we have

$$2\pi - \varepsilon < \int_K |\hat{\psi}_s(\xi)|^2 d\xi = \lim_{n \to \infty} \int_K \hat{\psi}_{t_n}(\xi) \overline{\hat{\psi}_s(\xi)} \, d\xi \leq$$

$$\limsup_{n\to\infty} \int_{\mathbf{R}} m_{t_n}\left(\frac{\xi}{2}+\pi\right) \hat{\varphi}_{t_n}\left(\frac{\xi}{2}\right) m_s\left(\frac{\xi}{2}+\pi\right) \hat{\varphi}_s\left(\frac{\xi}{2}\right) d\xi \le 2\pi - \varepsilon$$

which is a contradiction. Hence, from (38)

$$\|\hat{\psi}_t - \hat{\psi}_s\|_2^2 = \|\hat{\psi}_t\|_2^2 + \|\hat{\psi}_s\|_2^2 - \langle \hat{\psi}_t, \hat{\psi}_s\rangle - \langle \hat{\psi}_s, \hat{\psi}_t\rangle \longrightarrow 4\pi - 2\pi - 2\pi = 0$$

as $t \longrightarrow s$. This establishes the desired continuity of the path we constructed that connects ψ_0 to ψ_1 and Theorem IV is proved.

6 Further observations and description of the consortium

The results in this paper have been obtained in a genuine collaboration by members of the Wutam Consortium. There are and will be articles that elaborate some of the results announced here. We emphasize, however, that in this and subsequent reports by the Wutam Consortium, this collaboration will continue and each member of the consortium should consider these articles as part of their original scientific contribution.

Perhaps the first results on the connectivity of wavelets are contained in a paper by Aline Bonami, Silvain Durand and Guido Weiss [2]. These authors restricted their attention to wavelets produced by very smooth filters. The filters m they considered are elements of an infinite dimensional C^∞ manifold of 2π periodic functions m satisfying equality (15) and $m(0) = 1$. Their construction of a path joining two such filters is very different from the two presented in this paper. The continuity of the path is defined in terms of the topology introduced on the manifold. This does produce a continuous path of very smooth Fourier transforms of scaling functions, where the topology of the image is defined in terms of the L^2 norms of the functions involved and their derivatives. There is, however, a very important (and somewhat subtle) difference when all this is transferred to the Fourier transforms of wavelets defined in terms of the filters and scaling functions that have been constructed. There is a topological impediment that makes it impossible, in general, to obtain a corresponding continuous path of wavelets. In order to obtain an insight into this situation let us examine equality (5). The unimodular function $s(\xi)$, in this situation, must also be smooth and, hence, must belong to a homotopy class corresponding to an integer, the "winding number" of the path that has values in the circle of radius 1 about the origin in the complex plane. One can show that if we have a continuous path of wavelets, in the smooth sense we are considering, the corresponding path $t \longrightarrow s_t(\xi)$, $t \in [0,1]$, must be a jointly continuous function in the variables t and ξ. Thus, if s_0 and s_1 belong to different homotopy classes, they cannot be connected by an arc $t \longrightarrow s_t$ having this joint continuity property. A simple example of this situation is provided by trying to join the Haar wavelet ψ_0 to its translate by 1, $\psi_1 = \psi_0(\cdot - 1)$. If $\hat{\psi}_0$ satisfies equality (4), then ψ_1 satisies (5) with $s(\xi) = e^{-i\xi}$. Since this last function and the function that is identically equal to 1 are not in the same homotopy class, there is no path, of the type we described, that joins them. These questions, as well as an extension of the results by Bonami, Durand and Weiss can be found in the Ph.D. thesis of Gustavo Garrigós, a student of Weiss at Washington University.

Another interesting connectivity result was obtained by Darrin Speegle. He showed (in his thesis as a student at Texas A&M) that all *Minimally Supported Frequency* (MSF) wavelets are connected. These are the wavelets whose Fourier transform has an absolute value that equals the characteristic function of a set $K \subseteq \mathbf{R}$. Such a set must have measure equal to 2π. The support of the Fourier transform of any wavelet cannot have measure smaller than 2π; this is the motivation for so naming the above class. The MSF wavelets are of interest for many reasons. The well known Journé wavelet (see page 64 of [6]) was the first example of a non-MRA wavelet; it is an MSF wavelet. The Shannon wavelet is another example of a member of this class; it is, of course, an MRA wavelet. Auscher [1] and Lemarié [8] have, independently, shown that if one makes very mild assumptions about the Fourier transform of a wavelet (continuity and a decrease at ∞) it must be an MRA wavelet. These facts tend to indicate that "most" wavelets are either MRA wavelets or MSF wavelets. Thus, if one joins Theorem IV with Speegle's result it is not unreasonable to conjecture that the class of all wavelets is connected. We shall return to these considerations below. An elegant proof of the connectivity of a class of MSF wavelets can be found in [7]. It should also be noted that some notions of the connectivity of wavelets formed some of the motivation for the results obtained in [5].

The notion of a wavelet multiplier function were introduced in a paper of X. Dai, Qing Gu, David Larson and R. Liang [3] and in the Memoir [5] by Dai and Larson. The proof we presented of the characterization of these functions given in Theorem II is a variant of the one first produced by these four authors. The consideration of the other multipliers considered in Theorem III is naturally motivated by the study of wavelet multipliers. Further results related to these multipliers, as well as the structure of the classes $\mathcal{M}_{\psi_0}$, $\mathcal{W}_{\psi_0}$ and $\mathcal{S}_{\psi_0}$ can be found in a paper by Papadakis, Šikić and Weiss ([9]). We have already mentioned the fact that the equality $\mathcal{M}_\psi = \mathcal{W}_\psi$ may be false when ψ is not an MRA wavelet. Gu has obtained an example of two wavelets ψ_1, ψ_2 such that $\hat{\psi}_1 \neq \nu\hat{\psi}_2$ for all wavelet multiplier functions ν. In order to put these facts into some sort of perspective we first remind the reader that if ψ is an MSF wavelet, then $e^{i\lambda(\xi)}|\hat{\psi}(\xi)|$ is the Fourier transform of a wavelet for all real valued measurable functions $\lambda(\xi)$ (see the "remark" on pages 349-50 of [6]). Thus, $\mathcal{M}_\psi = \mathcal{W}_\psi$ when ψ is an MSF wavelet. In this paper we show that this equality holds for all MRA wavelets. These observations should be kept in mind in connection with our statement that "most wavelets are MSF or MRA wavelets;" Gu's example shows that there are other wavelets. At present, it is not clear where they "fit" with respect to the general connectivity question.

This particular exposition was written by Weiss who owes very special thanks to Garrigós, Paluszyński and Šikić for many discussions concerning the form that this exposition should take. They also read this manuscript as it was evolving and made several very appropriate suggestions and corrections. Theorems I and III will be elaborated in the collaboration [9]. The elegant argument used in the proof of Lemma 2.1 is mainly due to three students who took a course in the theory of wavelets given by Weiss: Marcin Bownik, Wojtek Czaja and Ziemovit Rzeszotnik. The proof of the lemma was assigned as an exercise in the course; their solution was certainly better than Weiss' original one. One can see from this solution the "inherent" discontinuity of the solution, $t(\xi)$ of the functional equation (10). This is related to the homotopy impediment we discussed four paragraphs above.

We shall now list the members of the Wutam Consortium in two groups, the one associated with Texas $A\&M$ and the one associated with Washington University. We shall give the present affiliation of each individual.

From Texas A&M

Xingde Dai, University of North Carolina at Charlotte;

Qing Gu, Texas A&M University;
Deguang Han, Texas A&M University;
David Larson, Texas A&M University;
Rufeng Liang, University of North Carolina at Charlotte;
Shijin Lu, Texas A&M University;
Darrin Speegle, St. Louis University.

From Washington University

Gustavo Garrigós, Washington University;
Eugenio Hernández, Universidad Autónoma de Madrid;
Maciej Paluszyński, Wroclaw University;
Manos Papadakis, Washington University;
Hrvoje Šikić, Washington University and the University of Zagreb;
David Weiland, University of Texas;
Guido Weiss, Washington University.

Not all fourteen of these individuals were directly involved in the research we described in this paper. This is, however, the first of what we plan to be a series of expositions of the collaborative research conducted by this group. Other results have already been obtained and will be described in these future publications. Each member of the Consortium has already made a contribution to the series. As stated above, it is only fair that each one of these researchers be considered as a collaborator for this article. We believe that these results, that are so closely connected, are best presented as a whole, rather as a number of papers whose inter-relations are not clearly explained.

X.Dai and D.Speegle obtained their Ph.D. degrees at Texas A&M, R.Liang is a student of Dai, Q.Gu, D.Han, and S.Lu are students working with Prof.Larson at Texas A&M. G.Garrigós is a student at Washington University, Hernández, Paluszyński and Weiland were students of Prof.Weiss at Washington University, Papadakis and Šikić are visiting faculty members at Washington University.

References

[1] Auscher, P., *Solution of two problems on wavelets*, J. of Geometric Anal-

ysis, Vol. 5, No. 2, 1995.

[2] Bonami, A., Durand, S. and Weiss, G., *Wavelets obtained by continuous deformations of the Haar wavelet*, Revista Mat. Iberoamericana, Vol. 12, No. 1, (1996).

[3] Dai, X., Gu, Q., Larson, D. and Liang, R., *Wavelet multipliers and the phase of an orthonormal wavelet*, to appear.

[4] Dai, X., Han, D., Liang, R. and Lu, S., *Some connectivity properties of wavelets*, preprint.

[5] Dai, X. and Larson, D. *Wandering vectors for unitary systems and orthogonal wavelets*, to appear in The Memoirs of the AMS.

[6] Hernández, E. and Weiss, G., *A First Course on Wavelets*, CRC Press, Boca Raton, (1996).

[7] Ionascu, E., Larson, D. and Pearcy, C., *On wavelet sets*, to appear in the Journal of Fourier Series and Applications.

[8] Lemarié-Rieusset, P. G. *Sur l'existence des analyses multiresolution en theorie des ondelettes*, Revista Mat. Iberoamericana, Vol. 8, No. 3 (1992).

[9] Papadakis, M., Šikić, H. and Weiss, G., *Characterizations and structure of functions associated with wavelets*, preprint.

[10] Speegle, D., *The s-elementary wavelets are connected*, preprint (1997).

[11] Stein, E. M. and Weiss, G., *Introduction to Fourier analysis on Euclidean spaces*, Princeton University Press, Princeton (1971)

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The Journal of Fourier Analysis and Applications
Volume 3, Number 4, 1997

Wavelet Sets in $\mathbb{R}^n$

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ABSTRACT. *A congruency theorem is proven for an ordered pair of groups of homeomorphisms of a metric space satisfying an abstract dilation-translation relationship. A corollary is the existence of wavelet sets, and hence of single-function wavelets, for arbitrary expansive matrix dilations on $L^2(\mathbb{R}^n)$. Moreover, for any expansive matrix dilation, it is proven that there are sufficiently many wavelet sets to generate the Borel structure of $\mathbb{R}^n$.*

A dyadic orthonormal (or orthogonal) wavelet is a function $\psi \in L^2(\mathbb{R})$, (Lebesgue measure), with the property that the set

$$\{2^{\frac{n}{2}}\psi(2^n t - l) \colon n, l \in \mathbb{Z}\}$$

is an orthonormal basis for $L^2(\mathbb{R})$ (see [1, 2]). For certain measurable sets, E, the normalized characteristic function $\frac{1}{\sqrt{2\pi}}\chi_E$ is the Fourier transform of such a wavelet. There are several characterizations of such sets (see [3] chapt. 4, and independently [5]). In [3] they are called wavelet sets. In [5, 6, 7] they are the support sets of MSF (minimally supported frequency) wavelets.

Dilation factors on $\mathbb{R}$ other than 2 have been studied in the literature, and analogous wavelet sets corresponding to all dilations > 1 are known to exist ([3], Example 4.5, part 10). Matrix dilations (for real expansive matrices) on $\mathbb{R}^n$ have also been considered in the literature, usually for a "multi-" notion of wavelet. The translations involved are those along the coordinate axes. The purpose of this article is to prove a general-principle type of result that shows, as a corollary, that analogous wavelet sets exist (and are plentiful) for all such dilations. In particular, "single-function" wavelets always exist. This appears to be new. Theorem 1 seems to belong to the mathematics behind wavelet theory. For this reason we prove it in a more abstract setting than needed for our wavelet results. Essentially, it is a dual-dynamical system congruency principle. The general proof is no more difficult than that for $\mathbb{R}^n$.

We point out that the wavelets we obtain, which are analogs of Shannon's wavelet, need not satisfy the regularity properties often desired (see [8]) in applications.

Let X be a metric space, and let m be a σ-finite nonatomic Borel measure on X for which the measure of every open set is positive and for which bounded sets have finite measure. Let $\mathcal{T}$ and $\mathcal{D}$ be countable groups of homeomorphisms of X which map bounded sets to bounded sets and which are absolutely continuous in the sense that they map m-null sets to m-null sets. A countable group $\mathcal{G}$ of absolutely continuous Borel isomorphisms of X determines an equivalence relation on the family $\mathcal{B}$ of Borel sets of X in a natural way: E and F are *$\mathcal{G}$-congruent* (written $E \sim_{\mathcal{G}} F$) if

Acknowledgements and Notes. The first author is supported in part by AFOSR grant F49620-96-1-0481 and a grant from the University of North Carolina at Charlotte. He was a participant in Workshop in Linear Analysis and Probability, Texas A&M University.

The second author is supported in part by NSF Grant DMS-9401544.

The third author was a Graduate Research Assistant at Workshop in Linear Analysis and Probability, Texas A&M University.

ISSN 1069-5869

there are measurable partitions $\{E_g\colon g \in \mathcal{G}\}$ and $\{F_g\colon g \in \mathcal{G}\}$ of E and F, respectively, such that $F_g = g(E_g)$ for each $g \in \mathcal{G}$, modulo m-null sets.

If $r > 0$ and $y \in X$, we write $B_r(y) := \{x \in X\colon \|x - y\| < r\}$, and abbreviate $B_r := B_r(0)$.

We will say that $(\mathcal{D}, \mathcal{T})$ is an *abstract dilation-translation pair* if (1) for each bounded set E and each open set F there are elements $\delta \in \mathcal{D}$ and $\tau \in \mathcal{T}$ such that $\tau(E) \subseteq \delta(F)$, and (2) there is a fixed point θ for $\mathcal{D}$ in X which has the property that if N is any nhood of θ and E is any bounded set, there is an element $\delta \in \mathcal{D}$ such that $\delta(E) \subseteq N$.

Theorem 1.

Let $X, \mathcal{B}, m, \mathcal{D}, \mathcal{T}$ be as above, with $(\mathcal{D}, \mathcal{T})$ an abstract dilation-translation pair, and with θ the $\mathcal{D}$-fixed point as above. Let E and F be bounded measurable sets in X such that E contains a nhood of θ, and F has nonempty interior and is bounded away from θ. Then there is a measurable set $G \subseteq X$, contained in $\bigcup_{\delta\in\mathcal{D}} \delta(F)$, which is both $\mathcal{D}$-congruent to F and $\mathcal{T}$-congruent to E.

Proof. We will use the term "$\mathcal{D}$-dilate" to denote the image of a set Ω under an element of $\mathcal{D}$, and "$\mathcal{T}$-translate" for the image of Ω under an element of $\mathcal{T}$.

We will construct a disjoint family $\{G_{ij}\colon i \in \mathbb{N}, j \in \{1,2\}\}$ of measurable sets whose $\mathcal{D}$-dilates form a partition $\{F_{ij}\}$ of F and whose $\mathcal{T}$-translates form a partition $\{E_{ij}\}$ of E, modulo m-null sets. Then $G = \bigcup_{i,j} G_{ij}$ will clearly satisfy our requirements. The i^{th} induction step will consist of constructing G_{i1} and G_{i2}.

Let $\{\alpha_i\}$ and $\{\beta_i\}$ be sequences of positive constants decreasing to 0. Let $N_1 \subset E$ be a ball centered at θ with radius $< \alpha_1$ such that $m(E\backslash N_1) > 0$. Let $E_{11} = E\backslash N_1$.

Observe that we may choose $\delta_1 \in \mathcal{D}$, $\tau_1 \in \mathcal{T}$, so that $(\delta_1^{-1} \circ \tau_1)(E_{11})$ is a subset of F whose relative complement in F has a nonempty interior. This is possible because since the interior of F is nonempty, there is a δ_1-dilate of F which contains a ball large enough to contain some τ_1-translate of E with ample room left over. Now set $F_{11} := (\delta_1^{-1} \circ \tau_1)(E_{11})$. (In this context, clearly we may choose δ_1 and τ_1 such that, in addition, the τ_1-translate of E is disjoint from any prescribed bounded set — a fact that will be useful in the second and subsequent steps.)

Let $G_{11} := \tau_1(E_{11}) = \delta_1(F_{11})$. Since δ_1 is a homeomorphism of X which fixes θ, G_{11} is bounded away from θ since F_{11} is. Let F_{12} be a measurable subset of F of positive measure, disjoint from F_{11}, such that the difference $F\backslash(F_{11} \cup F_{12})$ has a nonempty interior and measure $< \beta_1$. Choose $\gamma_1 \in \mathcal{D}$ such that $\gamma_1(F_{12})$ is contained in N_1 and is disjoint from G_{11}. Set $E_{12} := \gamma_1(F_{12})$, and set $G_{12} := E_{12}$. The first step is complete.

For the second step, note that since F is bounded away from θ, $N_1\backslash E_{12}$ contains a ball N_2 centered at θ with radius $< \alpha_2$ such that $N_1\backslash(E_{12} \cup N_2)$ has positive measure. Let

$$E_{21} := N_1\backslash(E_{12} \cup N_2) = E\backslash(E_{11} \cup E_{12} \cup N_2).$$

Choose $\delta_2 \in \mathcal{D}$, $\tau_2 \in \mathcal{T}$, using similar reasoning to that used above, such that $(\delta_2^{-1} \circ \tau_2)(E_{21})$ is a subset of $F\backslash(F_{11} \cup F_{12})$ whose relative complement in $F\backslash(F_{11} \cup F_{12})$ has a nonempty interior, and for which $\tau_2(E_{21})$ is disjoint from G_{11} and G_{12}. Let $F_{21} := (\delta_2^{-1} \circ \tau_2)(E_{21})$, and let $G_{21} := \tau_2(E_{21})$.

Choose a measurable subset $F_{22} \subset F$ of positive measure disjoint from F_{11}, F_{12}, F_{21} such that $F\backslash(F_{11} \cup F_{12} \cup F_{21} \cup F_{22})$ has a nonempty interior and measure $< \beta_2$. Noting that G_{11}, G_{12}, G_{21} are bounded away from θ, choose $\gamma_2 \in \mathcal{D}$ such that $\gamma_2(F_{22})$ is contained in N_2 and is disjoint from G_{11}, G_{12}, G_{21}. Set $E_{22} := \gamma_2(F_{22})$, and let $G_{22} := E_{22}$.

Now proceed inductively, obtaining disjointed families of sets of positive measure $\{E_{ij}\}$ in E, $\{F_{ij}\}$ in F, and $\{G_{ij}\}$, such that

$$\begin{aligned} \tau_i^{-1}(G_{i1}) &= E_{i1}, G_{i2} = E_{i2}, \delta_i^{-1}(G_{i1}) = F_{i1}, \\ \gamma_i^{-1}(G_{i2}) &= F_{i2}, \text{ for } i = 1, 2, \ldots \text{ and } j = 1, 2. \end{aligned}$$

We have $E\backslash(\cup E_{ij}) = \{\theta\}$, a null set, since $\alpha_i \to 0$, and $F\backslash(\cup F_{ij})$ is a null set since $\beta_i \to 0$. Let $G = \cup G_{ij}$. Since $\delta_i, \gamma_i \in \mathcal{D}$ we have $G_{ij} \in F$ for all i, j. So $G \subseteq \bigcup_{\delta \in \mathcal{D}} \delta(F)$. The proof is complete.

□

Remark. Suppose K is any bounded set that is bounded away from θ (i.e., K is contained in an annulus centered at θ). Then the set G in Theorem 1 can be taken *disjoint* from K. This follows immediately from the way the sets G_{ij} in the proof are constructed. Moreover, for each n a disjoint n-tuple can be constructed, all of which satisfy the properties of G and K above. To see this, mimic the proof of Theorem 1, at each step constructing $G_{ij}^1, \dots, G_{ij}^n$ simultaneously, making sure that they are disjoint from each other and also from all of the previous G_{lk}^h that have been constructed to that point. This construction can easily be modified to yield an infinite pairwise disjoint family $\{G^k\}_{k=1}^{\infty}$. □

We will now relate Theorem 1 to wavelets.

Let $1 \leq m < \infty$, and let A be an $n \times n$ real matrix which is *expansive* (equivalently, all eigenvalues have modulus > 1 (see [9])). By a dilation-A orthonormal wavelet we mean a function $\psi \in L^2(\mathbb{R}^n)$ such that

$$(*) \qquad \{|\det(A)|^{\frac{n}{2}} \psi(A^n t - (l_1, l_2, \dots, l_n)^t \colon n, l \in \mathbb{Z}\},$$

where $t = (t_1, \dots, t_n)^t$, is an orthonormal basis for $L^2(\mathbb{R}^n; m)$. (Here m is product Lebesgue measure, and the superscript "t" means transpose.)

It is useful to introduce dilation and translation unitary operators. If $A \in M_n(\mathbb{R})$ is invertible (so in particular if A is expansive), then the operator defined by

$$(D_A f)(t) = |\det A|^{\frac{1}{2}} f(At),$$

$f \in L^2(\mathbb{R}^n)$, $t \in \mathbb{R}^n$, is unitary. For $1 \leq i \leq n$, let T_i be the unitary operator determined by translation by 1 in the i^{th} coordinate direction. The set $(*)$ is then

$$\{D_A^k T_1^{l_1} \dots T_n^{l_n} \psi \colon k, l_i \in \mathbb{Z}\}.$$

The term orthogonal wavelet has been extended in the literature to include a "multi" notion, which is an orthonormal p-tuple $(f_1, \dots, f_p)$ of functions in $L^2(\mathbb{R}^n)$, each of which separately generates an incomplete orthonormal set under the system of unitaries, and which together form an o.n. basis.

Let $\mathcal{F}$ be the Fourier-Plancherel transform on $L^2(\mathbb{R})$, normalized so it is a unitary transformation. For $f, g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$,

$$\mathcal{F}(f)(s) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ist} f(t) dt$$

and

$$\mathcal{F}^{-1}(g)(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ist} g(s) ds.$$

On $L^2(\mathbb{R}^n)$ the Fourier transform is

$$(\mathcal{F}f)(s) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i(s \circ t)} f(t) dm,$$

where $s \circ t$ denotes the real inner product. Write $\hat{f} = \mathcal{F}f$, and for $A \in B(\mathbb{R}^n)$ write $\hat{A} := \mathcal{F}A\mathcal{F}^{-1}$. We have $\widehat{D}_A = D_{(A^t)^{-1}} (= D_{A^t}^{-1} = D_{A^t}^*)$, where A^t is the transpose of A, and $\widehat{T_j} = M_{e^{-is_j}}$, the multiplication operator on $\mathbb{R}^n$ with symbol $f(s_1, \dots, s_n) = e^{-is_j}$.

By a *dilation-A wavelet set* we will mean a measurable subset of $\mathbb{R}^n$ (necessarily of finite measure) for which the inverse Fourier transform of $(m(E))^{-\frac{1}{2}}\chi_E$ is a dilation-A orthonormal wavelet.

We will say that measurable subsets H and K of $\mathbb{R}^n$ are *A-dilation congruent* if there exist measurable partitions $\{H_l\}$ of H and $\{K_l\}$ of K such that $K_l = A^l H_l$, $l \in \mathbb{Z}$, modulo Lebesgue null-sets. Write $H \sim_{\delta_A} K$. We will also say that E, F are *2π-translation congruent* (write this $E \sim_{\tau_{2\pi}} F$) if there exist measurable partitions $\{E_l\colon l = (l_1, \ldots, l_n) \in \mathbb{Z}^n\}$ of E and $\{F_l\colon l \in \mathbb{Z}^n\}$ of F such that $F_l = E_l + 2\pi l$, $l \in \mathbb{Z}^n$, modulo null sets. If W is a measurable subset of $\mathbb{R}^n$ which is 2π-translation congruent to the n-cube $E = [-\pi, \pi) \times \cdots \times [-\pi, \pi)$, it is clear from the exponential form of $\widehat{T}_j$ that $\{\widehat{T}_1^{l_1}\widehat{T}_2^{l_2}\ldots\widehat{T}_n^{l_n} \cdot (m(W))^{-\frac{1}{2}}\chi_W\colon (l_1, \ldots, l_n) \in \mathbb{Z}^n\}$ is an o.n. basis for $L^2(W)$.

If A is a strict dilation, so $\|A^{-1}\| < 1$, then $AB_1 \supseteq B_{\|A^{-1}\|^{-1}}$. It follows that if $F = AB_1 \backslash B_1$, then $\{A^k F\colon k \in \mathbb{Z}\}$ is a partition of $\mathbb{R}^n\backslash\{0\}$. If A is expansive, then A is *similar* to a strict dilation. Therefore, $A = TCT^{-1}$ for T a real invertible $n \times n$ matrix, and with $\|C^{-1}\| < 1$. If $F_A = T(CB_1\backslash B_1)$, then $\{A^k F_A\}_{k\in\mathbb{Z}}$ is a partition of $\mathbb{R}^n\backslash\{0\}$. Therefore, an expansive matrix has a measurable complete *wandering set* $F_A \subset \mathbb{R}^n$. It follows that $L^2(F_A)$, considered as a subspace of $L^2(\mathbb{R}^n)$, is a complete wandering subspace for D_A. That is, $L^2(\mathbb{R}^n)$ is the direct sum decomposition of the subspaces $\{D_A^k L^2(F_A)\}_{k\in\mathbb{Z}}$. Moreover, it is clear that any measurable set F' with $F' \sim_{\delta_A} F_A$ has this same property.

Corollary 1.

Let $1 \le n < \infty$ and let $A \in M_n(\mathbb{R})$ be expansive. There exist dilation-A wavelet sets.

Proof. Let $\mathcal{A}$ be the group of homeomorphisms of $\mathbb{R}^n$ generated by the map $x \to A^t x$. Let $\mathcal{T}$ be the group of homeomorphisms of $\mathbb{R}^n$ generated by the translations in each of the coordinate directions by the integral multiples of 2π. Then A-dilation-congruency means $\mathcal{A}$-congruency and 2π-translation-congruency means $\mathcal{T}$-congruency. Moreover, it is clear that $(\mathcal{A}, \mathcal{T})$ is an abstract dilation-translation pair on $\mathbb{R}^n$ in the sense of Theorem 1, with $\theta = 0$.

By Theorem 1, a measurable set W exists with $W \sim_{\delta_{A^t}} F_{A^t}$ and $W \sim_{\tau_{2\pi}} E$, where $E = [-\pi, \pi)^n$. As remarked above, the set $\{\widehat{T}_1^{l_1}\widehat{T}_2^{l_2}\ldots\widehat{T}_n^{l_n}\widehat{\psi}_W\colon l_j \in \mathbb{Z}, 1 \le j \le n\}$ is an o.n. basis for $L^2(W)$, (with $\widehat{\psi}_W = (m(W))^{-\frac{1}{2}}\chi_W$). Therefore, since $L^2(W)$, regarded as a subspace of $L^2(\mathbb{R}^n)$, is wandering for $\widehat{D} := \widehat{D}_A = D_{A^t}^{-1}$, the set

$$\{\widehat{D}^k\widehat{T}_1^{l_1}\ldots\widehat{T}_n^{l_n}\widehat{\psi}\colon k \in \mathbb{Z}, l_j \in \mathbb{Z}, 1 \le j \le n\}$$

is an o.n. basis for $L^2(\mathbb{R}^n)$, and W is a wavelet set for A. (Moreover, by Remark 5 it follows that there is a countably infinite pairwise disjoint family of such sets.) □

A *Hardy* dyadic orthonormal wavelet is a function $\psi \in L^2(\mathbb{R})$ for which $\{2^{\frac{n}{2}}\psi(2^n t - \ell)\colon n, l \in \mathbb{Z}\}$ is an o.n. basis for the Hardy space of L^2-functions f whose Fourier transform $\hat{f}$ has support contained in $[0, \infty)$. An example is $\widehat{\psi} = (2\pi)^{-\frac{1}{2}}\chi_{[2\pi,4\pi)}$. Therefore, $[2\pi, 4\pi)$ is a Hardy wavelet set. This idea can be generalized.

Corollary 2.

Let $A \in M_n(\mathbb{R})$ be expansive, and let $M \subseteq \mathbb{R}^n$ be a measurable set of positive measures that is stable under A^t in the sense that $A^t M = M$. Suppose $M \cap F_{A^t}$ has a nonempty interior. Then there exist measurable sets $W \subset M$ with the property that, if $\widehat{\psi}_W := (2\pi)^{-\frac{n}{2}}\chi_W$, then

$$\{D_A^k T_1^{l_1}\ldots T_n^{l_n}\psi_W\colon k, l_i \in \mathbb{Z}\}$$

is an orthonormal basis for $\mathcal{F}^{-1}(L^2(M))$.

(Wavelets of this type were studied in [4] for the dyadic, $n = 1$ case, where they were called *subspace wavelets*. The concept is that they are wavelets for proper subspaces of $L^2(\mathbb{R})$.)

Proof. Apply Theorem 1, with $F = M \cap F_{A^t}$ and $E = [-\pi, \pi) \times \cdots \times [-\pi, \pi)$, obtaining W with $W \sim_{\tau_{2\pi}} E$ and $W \sim_{\delta_{A^t}} F$. Since M is A^t-stable, $W \subset M$. Also, $\{(A^t)^k W: k \in \mathbb{Z}\}$ is a measurable partition of M. Therefore, an argument similar to that before shows that $\{\widehat{D}_A^k \widehat{T}_1^{l_1} \ldots \widehat{T}_n^{l_n} \widehat{\psi}_W: k, l_i \in \mathbb{Z}\}$ is an o.n. basis for $L^2(M)$. □

The following result points out that the set of wavelet sets for any dilation is large. We will call an orthonormal wavelet for a dilation-factor $a > 1$, $a \in \mathbb{R}$, an *a-adic* orthonormal wavelet.

Corollary 3.

Let $A \in M_n(\mathbb{R})$ be expansive. Every measurable subset of $\mathbb{R}^n$ is a countable union of intersections of pairs of dilation-A wavelet sets. The family of Borel dilation-A wavelet sets generates the Borel structure of $\mathbb{R}^n$.

Proof. We first prove the a-adic case. Let $a > 1$ be arbitrary. Let $d(\cdot)$ denote the projection map from $\mathbb{R}\backslash\{0\}$ onto $F = [-a, -1) \cup [1, a)$ determined by a-dilation, and let $t(\cdot)$ denote the projection map from $\mathbb{R}$ onto $E = [-\pi, \pi)$ determined by 2π-translation. That is, for $x \in \mathbb{R}\backslash\{0\}$, $d(x)$ is the unique a-dilate of x contained in F, and for $x \in \mathbb{R}$, $t(x)$ is the unique 2π-translate of x contained in E. Note that $E \sim_{\tau_{2\pi}} [0, 2\pi) \sim_{\tau_{2\pi}} ([-2\pi, \pi) \cup [\pi, 2\pi)$. Suppose K is a measurable set in $\mathbb{R}\backslash\{0\}$ for which the restrictions $d|_K$ and $t|_K$ are one-to-one. Let $E_0 = E\backslash t(K)$ and $F_0 = F\backslash d(K)$. If E_0 contains a nhood of 0 and F_0 has a nonempty interior then by Theorem 1 and Remark 5 there are disjoint measurable sets G_1, G_2 with $G_i \sim_{\tau_{2\pi}} E_0$ and $G_i \sim_{\delta_a} F_0$, $i = 1, 2$. (By the construction in the proof of Theorem 1 (and Remark 5) if K is Borel, then these can be taken Borel.) Let $W_i = K \cup G_i$. Then $W_i \sim_{\tau_{2\pi}} E$ and $W_i \sim_{\delta_a} F$. So each W_i is an a-adic wavelet set. We have $K = W_1 \cap W_2$. We will show that each measurable set $G \subseteq \mathbb{R}$ has a measurable partition $\{G_j\}_j$ where each G_j has the property of K.

Observe that if K has the property in the above paragraph, i.e., $d(\cdot)$ and $t(\cdot)$ are 1-1, E_0 contains a nhood of 0 and F_0 has nonempty interior then every subset of K also has the property.

Suppose $0 < \alpha < \beta$, and let $J = [\alpha, \beta]$. If $\beta - \alpha < 2\pi$ then $t|_J$ is 1-1, and if $\beta < a\alpha$ then $d|_J$ is 1-1. If, in addition, J contains no integral multiple of 2π, then J satisfies the property of K above. Let $\mathcal{J}_+$ be the set of all intervals $[\alpha, \beta]$ with $0 < \alpha < \beta$, $\beta < \min\{a\alpha, \alpha + 2\pi\}$, $[\alpha, \beta] \cap 2\pi\mathbb{Z} = \phi$, α and β rational. Observe that $\cup\{J: J \in \mathcal{J}_+\} = (0, \infty)\backslash 2\pi\mathbb{Z}$. Let $\mathcal{J}_- = \{[-\beta, -\alpha]: [\alpha, \beta] \in \mathcal{J}_+\}$, and $\mathcal{J} = \mathcal{J}_+ \cup \mathcal{J}_-$. Then $\bigcup\limits_{J \in \mathcal{J}} J = \mathbb{R}\backslash 2\pi\mathbb{Z}$. Let $J_1, J_2, \ldots$ be an enumeration of $\mathcal{J}$, and let $L_1 = J_1$, and

$$L_{j+1} = J_{j+1}\backslash(J_1 \cup \cdots \cup J_j) \quad \text{for} \quad j \geq 1.$$

Then $\{L_j: j \in \mathbb{N}\}$ is a measurable partition of $\mathbb{R}\backslash 2\pi\mathbb{Z}$.

Let $G \subseteq \mathbb{R}$ be a measurable set. Clearly we may assume $G \cap 2\pi\mathbb{Z} = \phi$. Let $G_j = G \cap L_j$. Then $\{G_j\}$ is a measurable partition of G satisfying our requirements. If G is Borel, then each G_j is Borel.

We adapt the previous proof to the general case. Replace F with F_{A^t}, E with the n-cube $[-\pi, \pi) \times \cdots \times [-\pi, \pi)$, and $d(\cdot)$ and $t(\cdot)$ with the corresponding projections from $\mathbb{R}^n\backslash\{0\}$ to F_{A^t} and from $\mathbb{R}^n$ to E, respectively. If $K \subset \mathbb{R}^n$ has the property in paragraph one relative to these, the same argument shows that K is the intersection of two dilation-A wavelet sets. The boundary ∂C of the n-cube $C = [0, 2\pi) \times \cdots \times [0, 2\pi)$ is an m-null set. Let $Q = \cup\{(\partial C) + 2\pi\ell: \ell \in \mathbb{Z}^{(n)}\}$. By construction ∂F_{A^t} is also an m-null set. If $J = B_r(y)$ is a ball in $\mathbb{R}^n$ contained in one of the annuli $(A^t)^\ell F_{A^t}$ and which is also bounded away from Q, then J satisfies the property of K. Let $\mathcal{J}$ be the set of all such balls that have rational center and radius. Enumerate $\mathcal{J}$, define L_j as above, and observe that $\{L_j: j \in \mathbb{N}\}$ is a partition of $\mathbb{R}^n$ modulo a null set. As above, if $G \subseteq \mathbb{R}^n$ is a measurable, the partition $\{G \cap L_j: j \in \mathbb{N}\}$ satisfies our requirements. □

Remark. Theorem 1 can be improved in several further ways.

- 1. It is not necessary that m be nonatomic in Theorem 1. All that is needed is that $\{\theta\}$ is not an atom for m.
- 2. The hypothesis that E contains a nhood of θ in Theorem 1 can be replaced with the hypothesis that for each $\epsilon > 0$ there exists $\delta \in \mathcal{D}$ such that $\delta(F) \subseteq E \cap B_\epsilon(\theta)$. If we let $\widetilde{F} = \cup\{\delta(F): \delta \in \mathcal{D}\} \cup \{\theta\}$, then this is equivalent to the requirement that E contain a subset of $\widetilde{F}$ which is a nhood of θ in the relative topology of $\widetilde{F}$ in X. Remark 5 generalizes as well.
- 3. Theorem 1 remains true, in the general form of Remark 5 and 1, 2 above, if we drop the hypotheses that E and F are bounded and F is bounded away from θ. To adapt the proof, write $E = \bigcup_{i=0}^{\infty} E_i$, $F = \bigcup_{i=0}^{\infty} F_i$, $\{E_i\}$, $\{F_i\}$ disjoint, bounded, F_i bounded away from θ, and such that E_0 and F_0 play the role of E, F in the proof of Theorem 1; so E_0 contains a nhood of θ and F_0 has a nonempty interior. Then, for $k \geq 1$, in the k^{th} induction step (in which G_{k1} and G_{k2} are constructed), replace E with $E_0 \cup E_1 \cup \cdots \cup E_k$ and F with $F_0 \cup F_1 \cup \cdots \cup F_k$. The proof, thus modified, is easily seen to be valid.

□

References

[1] Chui, C.K. (1992). *An Introduction to Wavelets.* Academic Press, New York.

[2] Daubechies, I. (1992). *Ten Lectures on Wavelets.* CBMS 61, SIAM.

[3] Dai, X. and Larson, D., Wandering vectors for unitary systems and orthogonal wavelets, *Memoirs A.M.S.,* to appear.

[4] Dai, X. and Lu, S. (1996). Wavelets in subspaces. *Mich. J. Math.* **43**, 81–98.

[5] Fang, X. and Wang, X. (1996). Construction of minimally-supported-frequencies wavelets. *J. Fourier Analysis and Applications.* 2, 315–327.

[6] Hernandez, E., Wang, X., and G. Weiss (1996). Smoothing minimally supported frequency (MSF) wavelets: Part I. *J. Fourier Analysis and Applications.* 2, 329–340.

[7] Hernandez, E., Wang, X., and G. Weiss, Smoothing minimally supported frequency (MSF) wavelets: Part II. *J. Fourier Analysis and Applications,* to appear.

[8] Meyer, Y. (1992). *Wavelets and Operators*, Cambridge Studies in Advanced Mathematics. 37.

[9] Robinson, C. (1995). *Dynamical Systems.* CRC Press, Boca Raton, FL.

[10] Speegle, D., The s-elementary wavelets are connected, preprint.

Received May 20, 1996
Revised received December 20, 1996

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WAVELET SETS IN $\mathbb{R}^n$ II

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Abstract

This article concerns a functional analytic approach to certain aspects of the theory of orthonormal wavelets and represents some of the pure mathematics underlying wavelet theory. Wavelet sets have been useful in operator-theoretic interpolation of wavelets and in smoothing techniques for wavelets, as well as for pointing out the existence of single function wavelets in higher dimensions. We give some new concrete examples of wavelet sets in the plane and in $\mathbb{R}^n$, both for dilation factor 2 and for certain other expansive dilation matrices. We also review the state of the subject since our first paper, including some examples that have been worked out by others. We give a proof of the existence of subspace wavelets corresponding to dilation factor 2 for arbitrary dilation-invariant subsets of $\mathbb{R}^n$ of positive Lebesgue measure. This theorem is new to the literature and has not been published elsewhere.

1991 Mathematics Subject Classification: 47D25, 47N40, 46N99.

Key words and phrases: wavelet, wavelet set, operator, subspace wavelet.

[*]Supported in part by NSF Grant DMS-9401544.

[**]Supported in part by AFOSR grant F49620-96-1-0481 and a grant from the University of North Carolina at Charlotte.

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1 Introduction

In the article [4] the authors proved the existence of single-function orthonormal wavelets in $L^2(\mathbb{R}^n)$ for $n \geq 2$. If A is any real expansive $n \times n$ matrix (equivalently, all eigenvalues of A are required to have absolute value > 1), then the main result of [4] shows that orthonormal wavelets exist for the dilation factor A. That is, there is a single function $\psi \in L^2(\mathbb{R}^n)$ for which

$$\{|\det A|^{\frac{m}{2}}\psi(A^m t - \ell) \colon m \in \mathbb{Z}, \ell \in \mathbb{Z}^{(n)}\} \tag{1.1}$$

is an orthonormal basis of $L^2(\mathbb{R}^n)$. This was apparently somewhat of a surprise, because prior to this it was suspected by several researchers that functions ψ satisfying (1.1) did not exist, even for the special case where $A = 2$ (two times the $n \times n$ identity matrix). This latter is known as the dyadic case, and it has been regarded by many to be the case of maximum interest since it is strictly analogous to the one-dimensional dyadic case.

The wavelets considered in [4] have the form

$$\mathcal{F}^{-1}\left(\frac{1}{\sqrt{m(E)}} \cdot \chi_E\right)$$

for some measurable subset of $\mathbb{R}^n$, where $\mathcal{F}$ is the n-dimensional Fourier transform on $L^2(\mathbb{R}^n)$. (See section 3.) These are called MSF (*minimally supported frequency*) wavelets in the literature (see [8]). The sets E which are the support sets of MSF wavelets are called *wavelet sets.* MSF wavelets can be regarded as generalizations of Shannon's wavelet in the one-dimensional dyadic setting. The Shannon wavelet set is $[-2\pi, -\pi) \cup [\pi, 2\pi)$.

While the methods used in [4] were constructive they did not directly yield examples of an elegant nature. So no "concrete" examples were included in [4]. In response to this, Soardi and Weiland [18] constructed several examples in the plane of an interesting fractal-like character. Two others were independently constructed by Dai and Larson for inclusion in [5] in response to a kind suggestion of the referee. These were all for the case $A = 2$. One purpose of this article is to review these results, and in addition, to give some new examples for dilation factors different from 2. Another main purpose is to give a proof that "subspace" wavelets *always* exist for dilation factor 2. This theorem generalizes results in both [6] and [4].

Before continuing further with this article it is important that we take the opportunity to describe some of the *rationale* behind this work. Indeed, from an applications-oriented point of view, wavelet sets and their associated MSF wavelets are pathological in the sense that they have poor localization properties and have discontinuous Fourier transforms, and in addition, while many are affiliated with a multiresolution analysis (MRA) there are many that are not. Indeed, apart from the Shannon wavelet (which is MRA) the others of this type which appeared in the early literature, due to, for instance, Mallat and Journe, were given specifically to point out such instances of pathology. However, the role they play in our work is *not* one of pathology. In a real sense they play the role of *basic building blocks* in what is turning out to be a global approach to some of the pure mathematics underlying wavelet theory. They *can* lead to the construction of wavelets having good regularity and localization properties, in particular.

The basic idea in the role of wavelet sets in the operator-theoretic interpolation method given in [5] and discussed in [12] is just that, given a finite (or possibly even countably infinite) family of wavelet sets, there is a simple operator-theoretic criterion for when a given bounded measurable function with support contained in the *union* of the wavelet sets is the Fourier transform of a wavelet. In the finite family case, one simply writes down a bounded linear operator using a concrete formula in terms of the values of the given function on the individual wavelet sets. The function is the Fourier transform of an *orthonormal wavelet* if and only if the operator that is constructed is *unitary,* of a *Riesz wavelet* if and only if the operator is *invertible*, and of a *frame wavelet* if and only if the operator is a *co-isometry*. In many cases it is exceptionally difficult to analyze these properties of the operator. But in special cases, which are more common than might be expected, this can be done using operator theory and methods of C^*-algebra. This leads to new families of wavelets, and in some cases, famillies with good properties. In particular. in Chapter 5 of [5] it was shown that the classical Meyer-Lemarié class of orthonormal wavelets could be derived in this way by interpolating between two MSF wavelets in the class.

Another completely independent way of using wavelet sets and MSF wavelets as basic building blocks in a unified wavelet theory is given in the series [7, 9, 10] by Fang, Hernandez, Wang and Weiss, and is also described in the recent book [8] by Hernandez and Weiss, in which they developed techniques of *smoothing* appropriate MSF wavelets to obtain wavelets with better regularity properties. They showed, in particular, that the Meyer-Lemarié class could be derived by smoothing the Shannon wavelet in this way.

In concluding this section, we note that in [1] Auscher showed, in particular, that no single-function orthonormal wavelet for dilation factor $A = 2$ in dimension greater than one could be MRA or could have a Fourier transform that satisfied reasonable regularity hypotheses. Thus MSF wavelets supported on wavelet sets such as in Figure 1 and Figure 2 can not be completely smoothed. However, these sets are still useful in the theory because wavelet sets, and MSF wavelets, offer, due to their simplicity, a large number of concrete examples in the mathematical theory of wavelets that are convenient in our operator-theoretic approach for hands-on computation in testing hypotheses. As such, they have been especially good for the purpose of developing intuition concerning theorems and problems. Moreover, we have recently been able to show that single-function MRA wavelets and smoothing techniques do, in fact, exist in higher dimensions for certain matrix dilation factors other than $A = 2$ (the pure dyadic case). The interested reader can verify that, in fact, Figure 9 represents an MRA wavelet set in the plane. Further work in this direction will be the subject of a subsequent paper, and so will not be reported on in the present article.

2 Examples

The proof of the existence of wavelet sets in the paper [4], while constructive, yields wavelet sets which are unbounded, have infinitely many components, and no symmetry. In this section, we will give several examples of wavelet sets which are more aesthetically pleasing as well as having more potential for application. The first two are for the matrix $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ and are the ones given in section 6.6 of [5]. Example 2.1 is the same as Example 6.6.1 in [5], except for a larger diagram (Figure 1). The second is Example 6.6.2 in [5], but the diagram (Figure 2) is new: no diagram was given for this example in [5]. These wavelets are of the type in [18]; that is, in some sense, the examples consist of infinitely many pieces pasted together in a clever manner. We encourage the reader to consult the interesting article [18] for more wavelet sets and diagrams of this type.

It came as somewhat of a surprise that if we allow the matrix A to have a rotational component, then we can construct wavelet sets which have additional nice properties. For example, we can construct wavelet sets which are the union of two convex bodies, a wavelet set which has only one component, and even a wavelet set which is a square. These are Examples 2.3 and 2.4 which are taken from the Ph. D. thesis of the third author [16], and Example 2.5 which was obtained by the third author and Qing Gu, who kindly allowed us to include it in this article. We thank him.

We recall here the following characterization of wavelet sets. This was independently obtained by Fang and Wang [7] and by Dai and Larson [5] for the one-dimensional case, and is contained in [4] and described in [18] for the n-dimensional case.

Given a linear isomorphism $A : \mathbb{R}^n \to \mathbb{R}^n$, a measurable set $W \subset \mathbb{R}^n$ is a wavelet set with respect to A if and only if

$$\{A^i(W) : i \in \mathbb{Z}\} \quad \text{and} \quad \{W + 2\pi k : k \in \mathbb{Z}^n\}$$

are both partitions of $\mathbb{R}^n$. That is, if and only if the set W tiles $\mathbb{R}^n$ by translations and by dilations.

The way that the sets in Figure 1 and Figure 2 were constructed was by starting with a set E which tiles $\mathbb{R}^2$ by translations, then by translating pieces of E so that the resulting set tiles $\mathbb{R}^2$ by dilations. The construction of wavelet sets with respect to $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ in [5], [18]

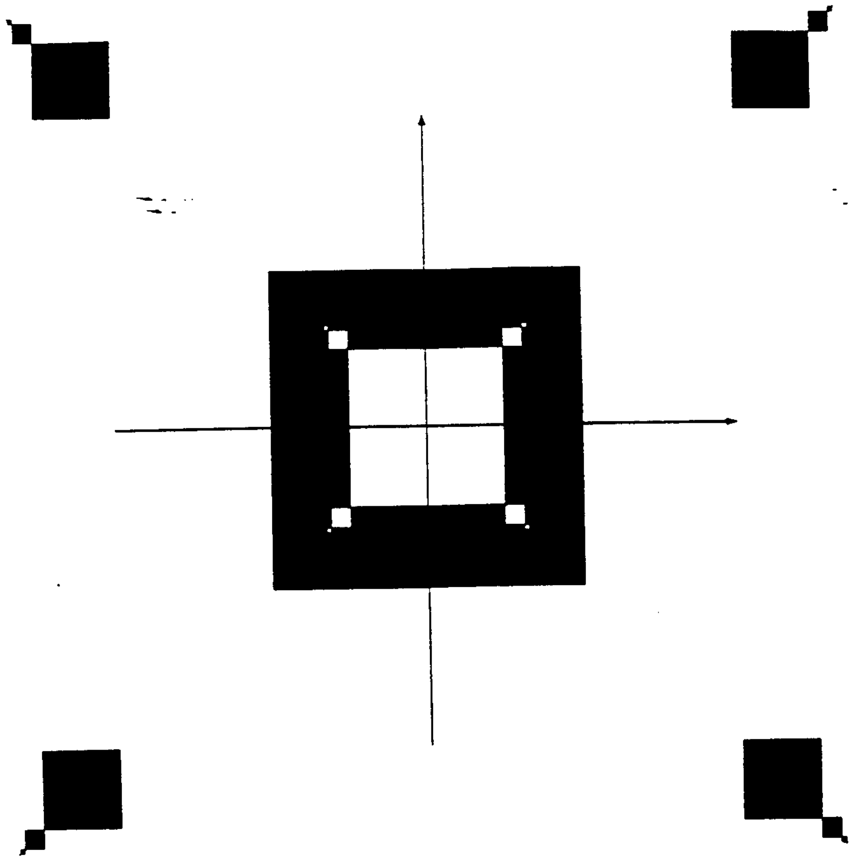

Figure 1. Wavelet set W_1 in $\mathbb{R}^2$

and [4] were all constrcuted using some type of iterative scheme. Such schemes yield wavelet sets which, in some sense, consist of infinitely many pieces. However, it is interesting to note that the example given in Figure 2 consists of only three connected components. The authors do not know if it is possible to construct a wavelet set with respect to 2I (two times the identity matrix) which consists of only one connected component. (For $n = 1$, there clearly is no such wavelet set, and it seems unlikely that there is one for $n = 2$. For other n, it is not clear to the authors what the conjecture should be.) Also, the sets in Figures 1 and 2 are bounded.

Example 2.1 The four corners set. (Figure 1.)

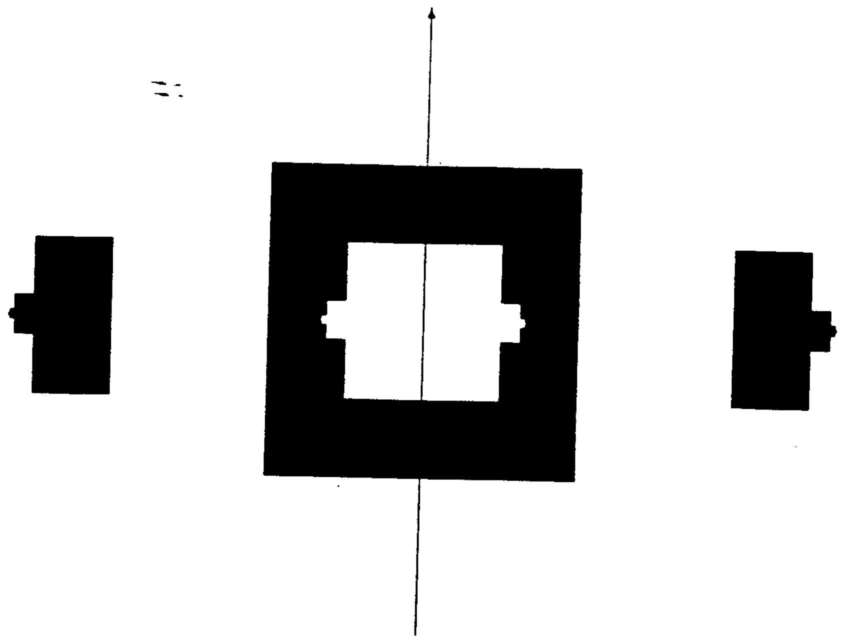

Figure 2. Wavelet set W_2 in $\mathbb{R}^2$

The diagram (Figure 1) for this was given in [5]. The proof we sketch is different and more intuitive, although less formal. Let W_{11} be the portion of W_1 contained in the first quadrant. We will describe W_{11} in detail, and the rest will be obvious by symmetry.

Define an increasing sequence of numbers by $d_0 := 0$, and $d_n := (1 + 4^{-1} + \cdots + 4^{-n+1})\frac{\pi}{2}$ for $n \geq 1$. Then, $d_{n+1} := d_n + 4^{-n}\frac{\pi}{2}$ and $d_n = \frac{2}{3}(1 - 4^{-n})\pi$ for $n \geq 0$. Observe that for $n \geq 0$ we have $d_n + 2\pi = 4d_{n+1}$. Now for $n \geq 0$, consider the sequence of squares

$$S_n := [d_n, d_{n+1}) \times [d_n, d_{n+1}).$$

In particular we have $S_0 = [0, \frac{\pi}{2}) \times [0, \frac{\pi}{2})$ and $S_1 = [\frac{\pi}{2}, \frac{5\pi}{8}) \times [\frac{\pi}{2}, \frac{5\pi}{8})$. It follows that $S_n + (2\pi, 2\pi) = 4S_{n+1}$ for all n. Let $S := \cup_{n=0}^{\infty} S_n$. Then,

$$S + (2\pi, 2\pi) = 4(\cup_{n=1}^{\infty} S_n).$$

Note that S is contained in the square $[0, \pi) \times [0, \pi)$. The difference set

$$[0, \pi) \times [0, \pi) \setminus S$$

is the part of W_{11} inside $[0, \pi) \times [0, \pi)$, and the translate $(S + (2\pi, 2\pi))$ is the part of W_{11} outside $[0, \pi) \times [0, \pi)$. See Figure 1. That is,

$$W_{11} = ([0, \pi) \times [0, \pi) \setminus S) \cup (S + (2\pi, 2\pi)).$$

By our construction, W_{11} is 2π-translation congruent to $[0, \pi) \times [0, \pi)$. Moreover, since

$$\frac{1}{4}(S + (2\pi, 2\pi)) = \cup_{n=1}^{\infty} S_n = S \setminus S_0$$

the set W_{11} is also 2-dilation congruent to the set $T_1 := [0, \pi) \times [0, \pi) \setminus [0, \frac{\pi}{2}) \times [0, \frac{\pi}{2})$.

Now, construct W_{12}, W_{13}, W_{14} and T_2, T_3, T_4, in the quadrants 2, 3 and 4 respectively, by symmetry. Then, $W_1 = \cup_{i=1}^{4} W_{1i}$. Let $T := \cup_{i=1}^{4} T_i = [-\pi, \pi) \times [-\pi, \pi) \setminus [-\frac{\pi}{2}, \frac{\pi}{2}) \times [-\frac{\pi}{2}, \frac{\pi}{2})$. Then W_1 is 2π-translation congruent to the square $[-\pi, \pi) \times [-\pi, \pi)$ which tiles $\mathbb{R}^2$ by 2π-translation, and W_1 is also 2-dilation congruent to T, a square dyadic torus, which tiles $\mathbb{R}^2$ by 2-dilation. This proves that W_1 is a dyadic wavelet set. ■

Example 2.2 The wedding cake set. (Figure 2.)

This example was given in [5] as example 6.6.2, but without a diagram. It is a construction very similar to Example 2.1. The proof here is different from [5]. Let W_{2R} be the portion of W_2 contained in the right half-plane. The left half W_{2L} will be obtained by symmetry. The idea is to excise from the rectangle $[0, \pi) \times [-\pi, \pi)$ an infinite union of disjoint adjacent rectangles which are stacked (sidewise here) like the layers of a wedding cake, and to translate this to the right by 2π. With appropriate care in construction, this will yield a wavelet set.

Let d_n, $n \geq 0$, be the same sequence of numbers as in Example 2.1. For $n \geq 0$, consider the rectangles

$$G_n := [d_n, d_{n+1}) \times [-4^{-n} \cdot \frac{\pi}{2}, 4^{-n} \cdot \frac{\pi}{2}).$$

In particular, $G_0 = [0, \frac{\pi}{2}) \times [-\frac{\pi}{2}, \frac{\pi}{2})$ and $G_1 = [\frac{\pi}{2}, \frac{5\pi}{8}) \times [-\frac{\pi}{8}, \frac{\pi}{8})$. Then $G_n + (2\pi, 0) = 4G_{n+1}$ for all n. Let $G := \cup_{n=0}^{\infty} G_n$. Then

$$G + (2\pi, 0) = 4(\cup_{n=1}^{\infty} G_n).$$

Observe that G is contained in the rectangle $[0, \pi) \times [-\pi, \pi)$, and the difference set $([0, \pi) \times [-\pi, \pi) \setminus G)$ is the part of W_{2R} inside $[0, \pi) \times [-\pi, \pi)$. See Figure 2. Also, the translate $(G + (2\pi, 0))$ is the part of W_{2R} outside $[0, \pi) \times [-\pi, \pi)$. So

$$W_{2R} = ([0, \pi) \times [-\pi, \pi) \setminus G) \cup (G + (2\pi, 0)).$$

Then W_{2R} is 2π-translation congruent to $[0, \pi) \times [-\pi, \pi)$. And, since

$$\frac{1}{4}(G + (2\pi, 0)) = \cup_{n=1}^{\infty} G_n = G \setminus G_0$$

W_{2R} is also 2-dilation congruent to $T_R := [0, \pi) \times [-\pi, \pi) \setminus [0, \frac{\pi}{2}) \times [-\frac{\pi}{2}, \frac{\pi}{2})$. Now construct W_{2L} and T_L in the left half-plane symmetrically. Then $W_2 = W_{2R} \cup W_{2L}$. Note $T = T_R \cup T_L$ is the same square torus as in Example 2.1. As in Example 2.1, W_2 is 2π-translation congruent to $[-\pi, \pi) \times [-\pi, \pi)$ and 2-dilation congruent to T; hence, W_2 is a dyadic wavelet set. ■

In general, it is more difficult to construct a wavelet set directly, without using approximation techniques as in [4], [5] and [18]. However, if we allow a rotational component in our matrix, then we can directly construct some simple examples of wavelet sets. The way the following example was constructed was by trial and error. Many different tilings of $\mathbb{R}^2$ by dilations were considered before one was discovered which also tiles $\mathbb{R}^2$ by translations.

Example 2.3 A wavelet set in the plane with two convex components.

We define $S_1 := \text{conv}\{(\frac{4}{3}\pi, 0), (\frac{2}{3}\pi, 0), (0, -2\pi), (0, -\pi)\}$ and $S_2 := \text{conv}\{((-\frac{4}{3}\pi, 0), (-\frac{2}{3}\pi, 0),$
$(0, \pi), (0, 2\pi)\}$. Then $W_3 = S_1 \cup S_2$ is a wavelet set for the matrix

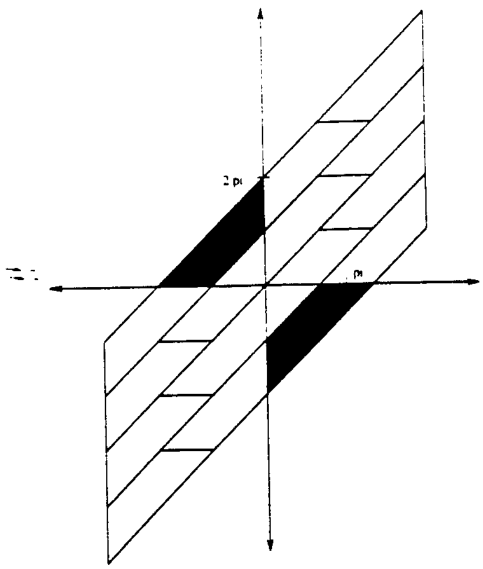

Figure 3. Translation tiling by the wavelet set W_3

$A = \begin{pmatrix} 0 & \frac{\sqrt{2}}{2} \\ \frac{-\sqrt{2}}{2} & 0 \end{pmatrix}$; that is, rotation by $\frac{\pi}{2}$ and dilation by $\sqrt{2}$. (See Figure 3 and Figure 4.)

The shaded portion of Figures 3 and 4 is the set $W_3 = S_1 \cup S_2$. The other polygons in Figure 3 give the tiling of the plane with respect to translation, while the polygons of Figure 4 give the tiling of the plane with respect to dilation by A. ■

We now give an example of an iterative scheme which yields a wavelet set with only one component.

Example 2.4 A wavelet set in the plane with one connected component.

Let $A = \begin{pmatrix} 0 & 2^{-1/4} \\ -2^{-1/4} & 0 \end{pmatrix}$; that is, rotation by $\frac{\pi}{2}$ clockwise and dilation by $2^{-1/4}$. Then, the set $E = \operatorname{conv}\{(-4\pi, 0), (-2\pi, 0), (0, 2\pi), (0, 4\pi)\}$ tiles $\mathbb{R}^2$ by dilations; the tiling is similar to that of Figure 4.

Now, it is clear that E does not tile $\mathbb{R}^2$ by translations, but the larger set $E_1 = \operatorname{conv}\{(-4\pi, 0), (-2\pi, 0), (0, 4\pi), (2\pi, 4\pi)\}$ does. Translating the piece of E_1 in Quadrant 1 to the left by 2π and combining it with E yields a set which still tiles $\mathbb{R}^2$ by translation, but its dilation by powers of A has *multiplicity*. That is, the dilations by powers of A fail to be disjoint. (See Figure 5.) Call this new set F.

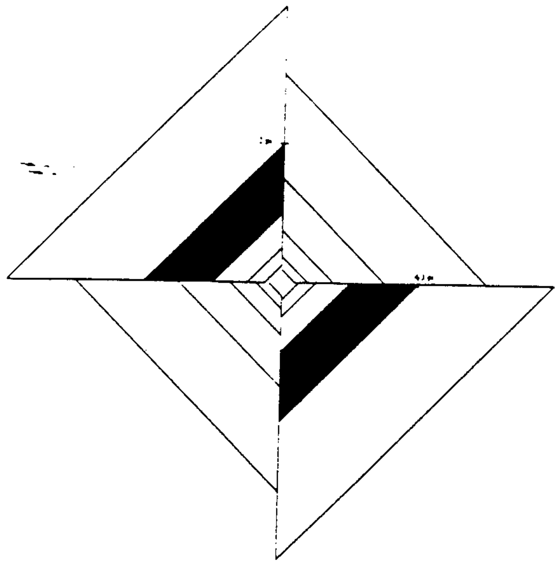

Figure 4. Dilation tiling by the wavelet set W_3

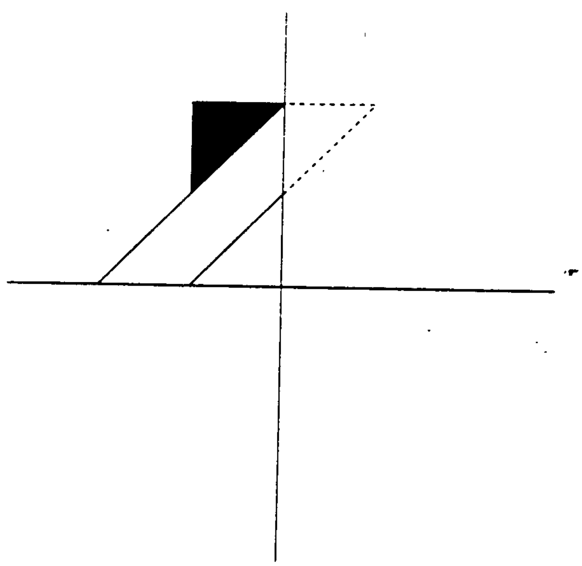

Figure 5. The set E_1

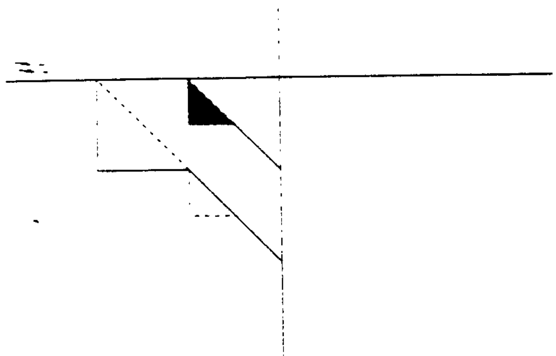

Figure 6. The set E_2.

Therefore, dilate the triangle shaded in Figure 5 by $\frac{1}{2}$ and remove it from F, and observe that the new set E_2 (Figure 6) tiles $\mathbb{R}^2$ by dilations. Note that E_2 does not tile $\mathbb{R}^2$ by translations; we are lacking the shaded piece indicated in Figure 6. So, we translate it to the left by 2π, and continue the process. For $0 \leq i < \infty$, we define rectangles Y_i by $Y_i = \text{conv}\{(2\pi(\frac{1}{2^i} - 1), 2\pi\frac{1}{2^i}), (2\pi(\frac{1}{2^i} - 1), 2\pi\frac{2}{2^i}), (2\pi(\frac{1}{2^i} - 2), 2\pi\frac{1}{2^i}), (2\pi(\frac{1}{2^i} - 2), 2\pi\frac{2}{2^i})\}$. Then, $W_4 = \cup_{i=0}^{\infty} Y_i$ is a wavelet set for the matrix A. The proof that this is a wavelet set is not hard and is outlined in Figure 7 and Figure 8.

The shaded portions of Figure 7 and Figure 8 correspond to the set W_4, while the other sets demonstrate the tiling with respect to dilation and translation respectively. While only the dilation tiling of quadrant II is given, the tilings of the other quadrants are appropriately scaled isomorphs of the given tiling. Likewise, only the translation tiling of the strip $\{(x, y) : 0 \leq y \leq 2\pi\}$ is given; the tilings of the other strips are similar. ∎

Example 2.5 A square wavelet set. (Due to Q. Gu and D. Speegle)

Let W_5 be the square $W_5 = \text{conv}\{(-\frac{8\pi}{2}, \frac{4\pi}{3}), (-\frac{8\pi}{3}, -\frac{2\pi}{3}), (-\frac{2\pi}{3}, \frac{4\pi}{3}), (-\frac{2\pi}{3}, -\frac{2\pi}{3})\}$, and let the dilation matrix be $A = \begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix}$. This

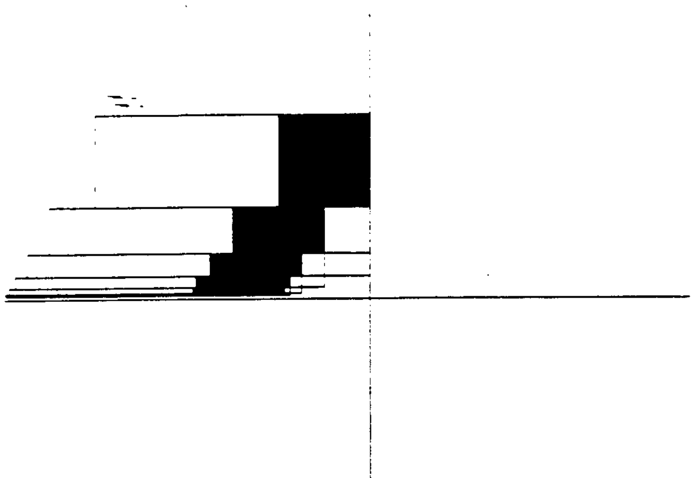

Figure 7. Dilation tiling by the one component wavelet set W_4

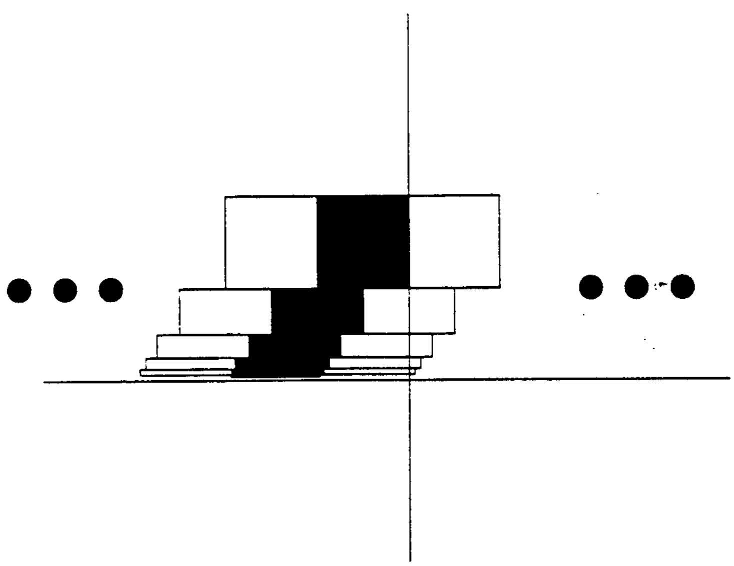

Figure 8. Translation tiling by the one component wavelet set W_4

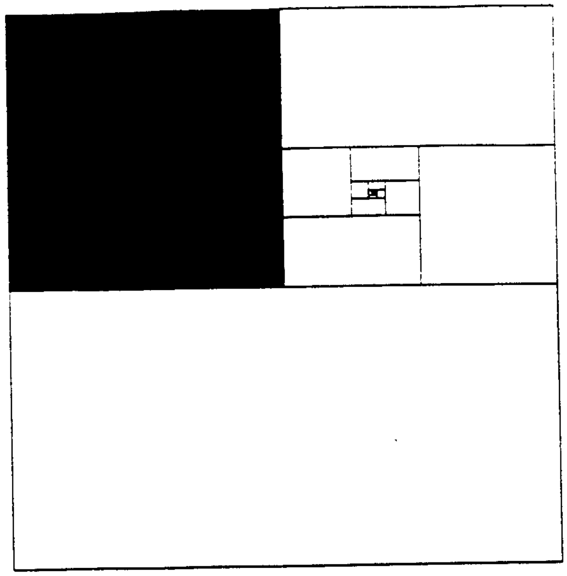

Figure 9. Dilation tiling by the square wavelet set W_5

square has length 2π, so it tiles $\mathbb{R}^2$ by translation. It is also tiles $\mathbb{R}^2$ by dilations, as is clear from Figure 9. ■

We note again that it is not known how nice wavelet sets in the plane with respect to the dilation matrix $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ can be. For example, it is not known whether there is a wavelet set with respect to $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ which is the union of finitely many disjoint rectangles. While this type of wavelet set may not have any applications, such simple wavelet sets have an inherent beauty. (Note that a square annulus tiles $\mathbb{R}^2$ by dilation, so dilations alone do not form an obstruction. Also, by the construction in [4], there is a wavelet set which is the union of infinitely many rectangles, each of which has positive distance from every other rectangle in the wavelet set.)

Some negative evidence is the following: Let $W \subset \mathbb{R}^n$ be the union of two compact, convex bodies in $\mathbb{R}^n$. Then, either W contains a neighborhood of the origin, or there is a pointed cone C of positive measure in $\mathbb{R}^n$ such that $C \cap W = \emptyset$. In the first case, W cannot be a wavelet set. In the second case, $\mathbb{R}^n \setminus C$ is invariant under the matrix 2I. So, $(\cup_{i=-\infty}^{\infty} 2^i(W)) \cap C = \emptyset$. Hence, W cannot be a dilation generator of $\mathbb{R}^n$; in particular, W is not a wavelet set.

3 Dyadic Subspace Wavelets

We require some material from [4]. The first is an abstract *dual-dynamical system congruency principal* which was our main tool in proving the existence of wavelet sets in dimensions $n \geq 2$.

Let X be a metric space, and let m be a σ-finite nonatomic Borel measure on X for which the measure of every open set is positive and for which bounded sets have finite measure. Let $\mathcal{T}$ and $\mathcal{D}$ be countable groups of homeomorphisms of X which map bounded sets to bounded sets and which are absolutely continuous in the sense that they map m-null sets to m-null sets. A countable group $\mathcal{G}$ of absolutely continuous Borel isomorphisms of X determines an equivalence relation on the family $\mathcal{B}$ of Borel sets of X in a natural way: E and F are *$\mathcal{G}$-congruent* (written $E \sim_{\mathcal{G}} F$) if there are measurable partitions $\{E_g\colon\ g \in \mathcal{G}\}$ and $\{F_g\colon\ g \in \mathcal{G}\}$ of E and F, respectively, such that $F_g = g(E_g)$ for each $g \in \mathcal{G}$, modulo m-null sets.

If $r > 0$ and $y \in X$ we write $B_r(y) := \{x \in X\colon\ \|x - y\| < r\}$, and abbreviate $B_r := B_r(0)$.

We will say that $(\mathcal{D}, \mathcal{T})$ is an *abstract dilation-translation pair* if (i) for each bounded set E and each open set F there are elements $\delta \in \mathcal{D}$ and $\tau \in \mathcal{T}$ such that $\tau(E) \subseteq \delta(F)$, and (ii) there is a fixed point θ for $\mathcal{D}$ in X which has the property that if N is any nhood of θ and E is any bounded set there is an element $\delta \in \mathcal{D}$ such that $\delta(E) \subseteq N$.

Theorem 3.1 *Let $X, \mathcal{B}, m, \mathcal{D}, \mathcal{T}$ be as above, with $(\mathcal{D}, \mathcal{T})$ an abstract dilation-translation pair, and with θ the $\mathcal{D}$-fixed point as above. Let E and F be bounded measurable sets in X such that E contains a nhood of θ, and F has nonempty interior and is bounded away from θ. Then there is a measurable set $G \subseteq X$, contained in $\bigcup_{\delta \in \mathcal{D}} \delta(F)$, which is both $\mathcal{D}$-congruent to F and $\mathcal{T}$-congruent to E.*

We will now relate Theorem 3.1 to wavelet sets.

Let $1 \leq m < \infty$, and let A be an $n \times n$ real matrix which is *expansive* (equivalently, all eigenvalues have modulus > 1 (cf. [15])). By a dilation-A orthonormal wavelet we mean a function $\psi \in L^2(\mathbb{R}^n)$

such that

$$\{|\det(A)|^{\frac{n}{2}}\psi(A^n t - (l_1, l_2, \ldots, l_n)^t) \colon n, l \in \mathbb{Z}\}, \tag{3.1}$$

where $t = (t_1, \ldots, t_n)^t$, is an orthonormal basis for $L^2(\mathbb{R}^n; m)$. (Here m is product Lebesgue measure, and the superscript "t" means transpose.)

By a *wandering set* for A we mean a measurable subset $S \subseteq \mathbb{R}^n$ which has the property that $\{A^n S \colon n \in \mathbb{Z}\}$ is a measurable partition of $\mathbb{R}^n$.

It is useful to introduce dilation and translation unitary operators. If $A \in M_n(\mathbb{R})$ is invertible (so in particular if A is expansive) then the operator defined by

$$(D_A f)(t) = |\det A|^{\frac{1}{2}} f(At),$$

$f \in L^2(\mathbb{R}^n)$, $t \in \mathbb{R}^n$, is unitary. For $1 \le i \le n$ let T_i be the unitary operator determined by translation by 1 in the i^{th} coordinate direction. The set (3.1) is then

$$\{D_A^k T_1^{l_1} \ldots T_n^{l_n} \psi \colon k, l_i \in \mathbb{Z}\}. \tag{3.2}$$

Let $\mathcal{F}$ be the Fourier-Plancherel transform on $L^2(\mathbb{R})$, normalized so it is a unitary transformation. For $f, g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$,

$$\mathcal{F}(f)(s) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ist} f(t) dt$$

and

$$\mathcal{F}^{-1}(g)(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ist} g(s) ds.$$

On $L^2(\mathbb{R}^n)$ the Fourier transform is

$$(\mathcal{F}f)(s) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i(s \circ t)} f(t) dm, \tag{3.3}$$

where $s \circ t$ denotes the real inner product. Write $\hat{f} = \mathcal{F}f$, and for $A \in B(\mathbb{R}^n)$ write $\widehat{D}_A := \mathcal{F} D_A \mathcal{F}^{-1}$. We have $\widehat{D}_A = D_{(A^t)^{-1}} (= D_{A^t}^{-1} = D_{A^t}^*)$, where A^t is the transpose of A, and $\widehat{T}_j = M_{e^{-is_j}}$, the multiplication operator on $\mathbb{R}^n$ with symbol $f(s_1, \ldots, s_n) = e^{-is_j}$.

By a *dilation-A wavelet set* we will mean a measurable subset of $\mathbb{R}^n$ (necessarily of finite measure) for which the inverse Fourier transform of $(m(E))^{-\frac{1}{2}}\chi_E$ is a dilation-A orthonormal wavelet.

We will say that measurable subsets H and K of $\mathbb{R}^n$ are *A-dilation congruent* if there exist measurable partitions $\{H_l\}$ of H and $\{K_l\}$ of K such that $K_l = A^l H_l$, $l \in \mathbb{Z}$, modulo Lebesgue null-sets. Write $H \sim_{\delta_A} K$. We will also say that E, F are *2π-translation congruent* (write this $E \sim_{\tau_{2\pi}} F$) if there exist measurable partitions $\{E_l\colon l = (l_1, \ldots, l_n) \in \mathbb{Z}^n\}$ of E and $\{F_l\colon l \in \mathbb{Z}^n\}$ of F such that $F_l = E_l + 2\pi l$, $l \in \mathbb{Z}^n$, modulo null sets. If W is a measurable subset of $\mathbb{R}^n$ which is 2π-translation congruent to the n-cube

$$E = [-\pi, \pi) \times \cdots \times [-\pi, \pi),$$

it is clear from the exponential form of $\widehat{T}_j$ that

$$\{\widehat{T}_1^{l_1}\widehat{T}_2^{l_2}\ldots\widehat{T}_n^{l_n} \cdot (m(W))^{-\frac{1}{2}}\chi_W\colon (l_1, \ldots, l_n) \in \mathbb{Z}^n\}$$

is an o.n. basis for $L^2(W)$.

If A is a strict dilation, so $\|A^{-1}\| < 1$, then $AB_1 \supseteq B_{\|A^{-1}\|^{-1}}$. It follows that if

$$F = AB_1 \backslash B_1,$$

then $\{A^k F\colon k \in \mathbb{Z}\}$ is a partition of $\mathbb{R}^n \backslash \{0\}$. If A is expansive then A is *similar* to a strict dilation. So $A = TCT^{-1}$ for T a real invertible $n \times n$ matrix, and with $\|C^{-1}\| < 1$. If

$$F_A = T(CB_1 \backslash B_1),$$

then $\{A^k F_A\}_{k \in \mathbb{Z}}$ is a partition of $\mathbb{R}^n \backslash \{0\}$. So an expansive matrix has a measurable complete *wandering set* $F_A \subset \mathbb{R}^n$. It follows that $L^2(F_A)$, considered as a subspace of $L^2(\mathbb{R}^n)$, is a complete wandering subspace for D_A. That is, $L^2(\mathbb{R}^n)$ is the direct sum decomposition of the subspaces

$$\{D_A^k L^2(F_A)\}_{k \in \mathbb{Z}}.$$

Moreover, it is clear that any measurable set F' with $F' \sim_{\delta_A} F_A$ has this same property.

The existence of wavelet sets is now nearly immediate from Theorem 3.1. To facilitate exposition we include the proof from [4].

Corollary 3.2 *Let $1 \leq n < \infty$ and let $A \in M_n(\mathbb{R})$ be expansive. There exist dilation-A wavelet sets.*

Proof. Let $\mathcal{A}$ be the group of homeomorphisms of $\mathbb{R}^n$ generated by the map $x \to A^t x$. Let $\mathcal{T}$ be the group of homeomorphisms of $\mathbb{R}^n$ generated by the translations in each of the coordinate directions by the integral multiples of 2π. Then A-dilation-congruency means $\mathcal{A}$-congruency and 2π-translation-congruency means $\mathcal{T}$-congruency. Moreover, it is clear that $(\mathcal{A}, \mathcal{T})$ is an abstract dilation-translation pair on $\mathbb{R}^n$ in the sense of Theorem 3.1, with $\theta = 0$.

By Theorem 3.1 a measurable set W exists with $W \sim_{\delta_{A^t}} F_{A^t}$ and $W \sim_{\tau_{2\pi}} E$, where $E = [-\pi, \pi)^n$. As remarked above, the set $\{\widehat{T}_1^{l_1}\widehat{T}_2^{l_2}\ldots\widehat{T}_n^{l_n}\widehat{\psi}_W\colon\ l_j \in \mathbb{Z},\ 1 \leq j \leq n\}$ is an o.n. basis for $L^2(W)$, (with $\widehat{\psi}_W = (m(W))^{-\frac{1}{2}}\chi_W$). So since $L^2(W)$, regarded as a subspace of $L^2(\mathbb{R}^n)$, is wandering for $\widehat{D} := \widehat{D}_A = D_{A^t}^{-1}$, the set

$$\{\widehat{D}^k\widehat{T}_1^{l_1}\ldots\widehat{T}_n^{l_n}\widehat{\psi}\colon\ k \in \mathbb{Z}, l_j \in \mathbb{Z}, 1 \leq j \leq n\}$$

is an o.n. basis for $L^2(\mathbb{R}^n)$, so W is a wavelet set for A. ■

The simplest type of *subspace* wavelet is a *Hardy* wavelet. We recall the definition. A *Hardy* dyadic orthonormal wavelet is a function $\psi \in L^2(\mathbb{R})$ for which $\{2^{\frac{n}{2}}\psi(2^n t - \ell)\colon\ n, l \in \mathbb{Z}\}$ is an o.n. basis for the Hardy space of L^2-functions f whose Fourier transform $\hat{f}$ has support contained in $[0, \infty)$. An example is $\widehat{\psi} = (2\pi)^{-\frac{1}{2}}\chi_{[2\pi,4\pi)}$. So $[2\pi, 4\pi)$ is a Hardy wavelet set. This idea can be generalized. To facilitate exposition we abstract the following result and its proof from [4].

Corollary 3.3 *Let $A \in M_n(\mathbb{R})$ be expansive, and let $M \subseteq \mathbb{R}^n$ be a measurable set of positive measure which is stable under A^t in the sense that $A^t M = M$. Suppose $M \cap F_{A^t}$ has nonempty interior. Then there exist measurable sets $W \subset M$ with the property that, if $\widehat{\psi}_W := (2\pi)^{-\frac{n}{2}}\chi_W$, then*

$$\{D_A^k T_1^{l_1}\ldots T_n^{l_n}\psi_W\colon\ k, l_i \in \mathbb{Z}\} \tag{3.4}$$

is an orthonormal basis for $\mathcal{F}^{-1}(L^2(M))$.

Proof. Apply Theorem 3.1, with $F = M \cap F_{A^t}$ and $E = [-\pi, \pi) \times \cdots \times [-\pi, \pi)$, obtaining W with $W \sim_{\tau_{2\pi}} E$ and $W \sim_{\delta_{A^t}} F$. Since

M is A^t-stable, $W \subset M$. Also, $\{(A^t)^k W\colon k \in \mathbb{Z}\}$ is a measurable partition of M. So an argument similar to that above shows that $\{\widehat{D}_A^k \widehat{T}_1^{l_1} \ldots \widehat{T}_n^{l_n} \widehat{\psi}_W\colon k, l_i \in \mathbb{Z}\}$ is an o.n. basis for $L^2(M)$. ■

Remark 3.4 Wavelets of the type in Corollary 3.3 were studied in [6] for the dyadic $n = 1$ case, where they were called *subspace wavelets*. The concept is that they are wavelets for proper subspaces of $L^2(\mathbb{R})$.

The new result that we wish to present is Theorem 3.7 below, which generalizes Corollary 3.3 in the special case of dilation matrix $A = 2I$ (the dyadic case) in the sense that the requirement that $M \cap F_{A^t}$ has nonempty interior is completely eliminated. So M can be any dilation stable measurable subset of $\mathbb{R}^n$ with positive measure. The "nonempty interior" condition can be replaced with Lebesgue density considerations. These are well-developed in the classical literature for scale factor 2 (that is, dilation factor 2), and readily extend to other scale factors (that is, scalar dilation factors). But there seem to be some real obstacles to extending this (density) theory to general expansive dilation factors. For this reason we elected not to include Theorem 3.7 in our initial paper [4]. We felt at that time that the result was not as general as we wanted. However, since then several of our colleagues have indicated to us that the dyadic case is the case of greatest current interest and have suggested that our result should be published. Thus we include it in this follow-up paper.. We state, and prove, our result only for the dyadic case. We note, however, that the proof adapts trivially to the arbitrary positive scalar dilation factor case.

Let E be a measurable set in $\mathbb{R}^n$ with $m(E) > 0$. Then E has a *point of density* y.

This means (cf. [14, p. 261]) that

$$\operatorname*{Lim}_{r \to 0} \frac{m(E \cap C_r(y))}{m(C_r(y))} = 1 \tag{3.4}$$

where $C_r(y)$ is the n-cube with center y, edge length $2r$, and edges parallel to the coordinate axes. Since the ratio of the measures of $C_r(y)$ and $B_r(y)$ is independent of r (where $B_r(y)$ is the ball, center y, radius r) it is easy to see that this is equivalent to

$$\operatorname*{Lim}_{r\to 0}\frac{m(E\cap B_r(y))}{m(B_r(y))}=1. \tag{3.5}$$

Hence

$$\operatorname*{Lim}_{k\to\infty}\frac{m(E\cap B_{2^{-k}}(y))}{m(B_{2^{-k}}(y))}=1. \tag{3.6}$$

Fix a positive integer p. Multiplying numerator and denominator by 2^{k+p} we have

$$\operatorname*{Lim}_{k\to\infty}\frac{m(2^{k+p}E\cap B_{2^p}(2^{k+p}y)}{m(B_{2^p}(2^{k+p}y))}=1, \tag{3.7}$$

or

$$\operatorname*{Lim}_{k\to\infty} m(2^{k+p}E\cap B_{2^p}(2^{k+p}y))=m(B_{2^p}). \tag{3.8}$$

(Where we write $B_r := B_r(0)$.) So, given $\epsilon>0$ there exists $k_0>0$ such that for $k>k_0$ we have

$$m(2^{k+p}E\cap B_{2^p}(2^{k+p}y))\geq(1-\epsilon)m(B_{2^p}). \tag{3.9}$$

Now suppose K is a prescribed bounded measurable set, and p is taken sufficiently large so that

$$2^p>\operatorname{diam}(K)+\frac{\sqrt{n}}{2}.$$

Then since for any $w\in\mathbb{R}^n$ we have $\operatorname{dist}(w,\mathbb{Z}^n)\leq\frac{\sqrt{n}}{2}$, it follows that there is an n-tuple of integers $z_k=(z_k(1),\ldots,z_k(n))\in\mathbb{Z}^n$ so that

$$K-z_k\subseteq B_{2^p}(2^{k+p}y).$$

Then for $k\geq k_0$,

$$\begin{aligned}
m(2^{k+p}E\cap(K-z_k) &\geq m(2^{k+p}E\cap B_{2^p}(2^{k+p}y))\\
&\quad -(m(B_{2^p}(2^{k+p}y)\backslash(K-z_k)))\\
&\quad > (1-\epsilon)m(B_{2^p})-m(B_{2^p})+m(K)\\
&= m(K)-\epsilon m(B_{2^p}).
\end{aligned} \tag{3.10}$$

This proves, with an obvious "ϵ" adjustment in the proof in case K is not bounded but has finite measure:

Proposition 3.5 *Let E and K be measurable subsets of $\mathbb{R}^n$, with $m(E) > 0$ and with $m(K) < \infty$. Then given $\epsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that for each $k \geq k_0$ there exists $z_k \in \mathbb{Z}^n$ such that*

$$m(2^k E \cap (K - z_k)) \geq (1 - \epsilon) m(K). \tag{3.11}$$

Moreover, if 0 is a point of density for E, then we may take $z_k = 0$ for all k.

Remark 3.6 It is clear that Proposition 3.5 remains valid with translation by points in $\mathbb{Z}^n$ replaced with translation by points in $2\pi \cdot \mathbb{Z}^n$. That is, with "z_k" in (3.11) replaced by "$2\pi z_k$". This is the form we will need.

To prove our main result we will appropriately modify the proof of Theorem 3.1 found in [4]. The reader will notice a similarity in proof, although significantly many more details are needed and the adaptation is not very straightforward. The interested reader may wish to directly compare the proofs to gain some insight.

Theorem 3.7 *Let n be a positive integer. Let $M \subseteq \mathbb{R}^n$ be a measurable set of positive measure with the property that $M = 2M$. Then there exists a measurable set $G \subset M$ with the property that, if $\widehat{\psi}_G := (2\pi)^{-\frac{n}{2}} \chi_G$, then*

$$\left\{ 2^{\frac{nm}{2}} \widehat{\psi}(2^m t - \ell) \colon \; m \in \mathbb{Z}, \ell \in \mathbb{Z}^{(n)} \right\} \tag{3.12}$$

is an orthonormal basis for $\mathcal{F}^{-1}(L^2(M))$.

Proof. First note that $\det(2I) = 2^n$, so (3.12) fits the form (1.1). Let $E = [-\pi, \pi]^{(n)}$ and let $T = B_2(0) \backslash B_1(0)$. Let $F = M \cap T$. Then $\{2^n F \colon \; m \in \mathbb{Z}\}$ is a measurable partition of M, and F has positive measure.

We will construct a disjoint family $\{G_{ij} \colon \; i \in \mathbb{N}, j \in \{1, 2,\}\}$ of measurable sets whose 2-dilates form a measurable partition $\{F_{ij}\}$ of F and whose translates by vectors in $2\pi \cdot \mathbb{Z}^{(n)}$ form a measurable partition of E. Then

$$G := \cup \{G_{ij} \colon \; i \in \mathbb{N}, j = 1, 2\}$$

will satisfy our requirements, in view of the exposition following Theorem 3.1. The i^{th} induction step will consist of constructing G_{i1} and G_{i2}.

Let $\{\alpha_i\}$ and $\{\beta_i\}$ be sequences of positive constants decreasing to 0 and with $\alpha_1 < \pi$. Let N_1 be a ball centered at 0 with radius $< \alpha_1$. Let $\widetilde{E}_{11} = E\backslash N_1$. Then $m(\widetilde{E}_{11}) > 0$. Let $\widetilde{F}_{11}$ be a measurable subset of F with measure strictly less than $m(F)$. By Proposition 3.5 and Remark 3.6, there exists $k_1 \in \mathbb{N}$ and $\ell_1 \in \mathbb{Z}^n$, so that

$$m(2^{k_1}\widetilde{F}_{11} \cap (\widetilde{E}_{11} - 2\pi\ell_1)) \geq \frac{1}{2}m(\widetilde{E}_{11}).$$

Let $G_{11} := 2^{k_1}\widetilde{F}_{11} \cap (\widetilde{E}_{11} - 2\pi\ell_1)$, let $E_{11} := G_{11} + 2\pi\ell_1$, and let

$$F_{11} := \widetilde{F}_{11} \cap 2^{-k_1}(E_{11} - 2\pi\ell_1).$$

Then $F_{11} \subseteq F$ and $m(F\backslash F_{11}) > 0$. Also $E_{11} \subseteq \widetilde{E}_{11}$, and

$$m(E_{11}) = m(G_{11}) \geq \frac{1}{2}m(\widetilde{E}_{11}).$$

Also $G_{11} = 2^{k_1}F_{11}$. Now choose $F_{12} \subseteq F$, disjoint from F_{11}, such that $F\backslash(F_{11} \cup F_{12})$ has positive measure $< \beta_1$. Choose $m_1 \in \mathbb{N}$ so that $2^{-m_1}F_{12}$ is contained in N_1 and is disjoint from G_{11}. (This is possible because G_{11} is bounded away from 0 since $\widetilde{F}_{11} \subseteq F$.) Set

$$G_{12} := E_{12} := 2^{-m_1}F_{12}.$$

The first step is complete.

For the second step, which is slightly more complicated than the first, note that since F is bounded away from $0, N_1\backslash E_{12}$ contains a ball N_2 centered at 0 with radius $< \alpha_2$ such that $N_1\backslash(E_{12} \cup N_2)$ has positive measure. Let

$$\widetilde{E}_{21} := E\backslash(E_{11} \cup E_{12} \cup N_2).$$

Then $m(\widetilde{E}_{21}) > 0$. Let $\widetilde{F}_{21}$ be a measurable subset of $F\backslash(F_{11} \cup F_{12})$ for which

$$m(F\backslash(F_{11} \cup F_{12} \cup \widetilde{F}_{21})) > 0.$$

Since G_{11} and G_{12} are bounded and F is bounded away from 0 there exists $\tilde{k}_2 \in \mathbb{N}$ with the property that $k \geq \tilde{k}_2 \Rightarrow 2^kF \cap (G_{11} \cup G_{12})$ is empty. Again by Proposition 3.5 and Remark 3.6, there exists $k_2 \in \mathbf{N}$ and $\ell_2 \in \mathbb{Z}^n$ with $k_2 \geq \tilde{k}_2$ so that

$$m(2^{k_2}\widetilde{F}_{21} \cap (\widetilde{E}_{21} - 2\pi\ell_2)) \geq \frac{1}{2}m(\widetilde{E}_{21}).$$

Let

$$G_{21} := 2^{k_2}\widetilde{F}_{21} \cap (\widetilde{E}_{21} - 2\pi\ell_2),$$

let $E_{21} := G_{21} + 2\pi\ell_2$, and let

$$F_{21} := \widetilde{F}_{21} \cap 2^{-k_2}(E_{21} - 2\pi\ell_2).$$

Then F_{21} is a subset of $F\backslash(F_{11} \cup F_{12})$ for which $F\backslash(F_{11} \cup F_{12} \cup F_{21})$ has positive measure. Also, $E_{21} \subseteq \widetilde{E}_{21}$, and

$$m(E_{21}) = m(G_{21}) \geq \frac{1}{2}m(\widetilde{E}_{21}).$$

Also $G_{21} = 2^{k_2}F_{21}$. The condition $k_2 \geq \tilde{k}_2$ implies G_{21} is disjoint from G_{11} and G_{12}. Choose a measurable subset $F_{22} \subset F$ of positive measure disjoint from F_{11}, F_{12}, F_{21} such that $F\backslash(F_{11} \cup F_{12} \cup F_{21} \cup F_{22})$ has positive measure $< \beta_2$. Noting that G_{11}, G_{12}, G_{21} are bounded away from 0, choose $m_2 \in \mathbf{N}$ such that $2^{-m_2}F_{22}$ is contained in N_2 and is disjoint from G_{11}, G_{12}, G_{21}. Set

$$G_{22} := E_{22} := 2^{-m_2}F_{22}.$$

Now proceed inductively, obtaining disjoint families of sets of positive measure $\{E_{ij}\}$ in E, $\{F_{ij}\}$ in F and $\{G_{ij}\}$, such that for $i = 1, 2, \ldots$ and $j = 1, 2$ we have

(i) $G_{i1} + 2\pi\ell_i = E_{i1}$,

(ii) $G_{i2} = E_{i2}$,

(iii) $2^{-k_i}G_{i1} = F_{i1}$,

(iv) $2^{m_i}G_{i2} = F_{i2}$,

(v) $m(F\backslash(F_{11} \cup F_{12} \cup \cdots \cup F_{i1} \cup F_{i2})) < \beta_i$, and

(vi) $$m(E_{i1}) \geq \frac{1}{2}m(E\backslash(E_{11} \cup E_{12} \cup \cdots \cup E_{i-1,1} \cup E_{i-1,2} \cup N_i))$$
$$\geq \frac{1}{2}m(E\backslash(E_{11} \cup E_{12} \cup \cdots \cup E_{i-1,1} \cup E_{i-1,2})) - \frac{1}{2}m(N_i)$$
where N_i is a ball centered at 0 with radius $< \alpha_i$.

Since $\beta_i \to 0$ item (v) implies that $F \backslash (\cup F_{ij})$ is a null set, and since $\alpha_i \to 0$ then (vi) implies that $E \backslash (\cup E_{ij})$ is a null set. Let

$$G = \cup\{G_{ij} \colon \ i = 1, 2, \ldots, j = 1, 2\}.$$

Then G is 2π-translation congruent to $E = [-\pi, \pi]^{(n)}$ by items (i) and (ii). It is also 2-dilation congruent to F by items (iii) and (iv), and thus the 2-dilates of G form a measurable partition of M. The argument in Corollary 3.3 now applies, concluding the proof. ■

References

[1] P. Auscher. Solutions of two problems on wavelets. J. of Geometric Analysis. 1995 (5), 181–236.

[2] C.K. Chui. *An Introduction to Wavelets*, Acad. Press, New York, 1992.

[3] I. Daubechies, *Ten Lectures on Wavelets*, CBMS 61, SIAM, 1992.

[4] X. Dai. D. Larson and D. Speegle, Wavelet sets in $\mathbb{R}^n$, J. Fourier Analysis sand Applications, to appear.

[5] X. Dai and D. Larson, Wandering vectors for unitary systems and orthogonal wavelets, Memoirs A.M.S., to appear.

[6] X. Dai and S. Lu, Wavelets in subspaces, Mich. J. Math. 43 (1996) 81–98.

[7] X. Fang and X. Wang, Construction of minimally-supported-frequencies wavelets, J. Fourier Analysis and Applications, 2 (1996), 315–327.

[8] E. Hernandez and G. Weiss, A first course in wavelets, CRC Press, Boca Raton, 1996.

[9] E. Hernandez, X. Wang and G. Weiss, Smoothing minimally supported frequency (MSF) wavelets: Part I, J. Fourier Analysis and Applications, 2 (1996), 329–340.

[10] E. Hernandez, X. Wang and G. Weiss, Smoothing minimally supported frequency (MSF) wavelets: Part II, J. Fourier Analysis and Applications, to appear.

[11] E. Ionascu, D. Larson and C. Pearcy, On the unitary systems affiliated with orthonormal wavelet theory in n-dimensions, preprint.

[12] D. Larson, von Neumann algebras and wavelets, Proceedings of NATO Advanced Study Institute on Operator Algebras and Applications, August 1996, to appear.

[13] Y. Meyer, *Wavelets and Operators*, Camb. Studies in Adv. Math., 37, 1992.

[14] I.P. Natanson, Theory of functions of a real variable, translation from Russian, F. Ungar, New York, 1961.

[15] C. Robinson, *Dynamical Systems*, CRC Press, Boca Raton, 1995.

[16] D. Speegle, Ph.D. Dissertation, Texas A&M University, 1997.

[17] D. Speegle, The s-elementary wavelets are connected, Proc. AMS, to appear.

[18] P. Soardi and D. Weiland, Single wavelets in n-dimensions, preprint.

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